Math-23324: Geometric Modeling

By

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0.1 Syllabus

Implicit and parametric forms of curve and surfaces, power basis form of curve, Bezier curve, rational Bezier curves, tensor product surfaces, introduction of B-Spline basis functions,B-spline basis function, derivatives of B-spline basis function, computational algorithms, introduction of B-Spline curve and surfaces,B-spline curve, derivatives of B-spline curve, B-spline surfaces, derivatives of B-spline surfaces, introduction of rational B-Spline curve and surfaces,NURBS curves, derivatives of NURBS curves, NURBS surfaces, derivatives of NURBS surfaces, introduction of approximation and interpolation, Lagrange form, Newton form, Hermite interpolation, Piecewise cubic Hermite interpolation, approximation, least squares fitting.

0.1.1 Recommended Books

- Gerald Farin, Curves and Surfaces for CAGD: A Practical Guide, 5th Edition, Amazon (2002)

- G. Farin, J. Hoschek and M. Sookim, Hand Book of Computer Aided Geometric Design, Elsevier Science (2002)

- David F. Rogers, Procedural Elements for Computer Graphics. McGraw-Hill Companies, Inc.(1998).

Contents

Chapter 1

Representations of the curves

Graphical representation of functions is called curves. There are three types of representations of curves.

- Explicit representation
- Implicit representation
- Parametric representation

Definition 1.0.1. Explicit representation:
In this representation, we represent the one variable in terms of another variable using a single valued function i.e. $y = f(x)$. Examples of explicit curves include:

A curve:	$f(x) = 3x + 6,$
An upper semi-circle:	$f(x) = +\sqrt{1 - x^2},$
A lower semi-circle:	$f(x) = -\sqrt{1 - x^2}.$

It is difficult to represent closed shapes by using explicit representation.

Definition 1.0.2. Implicit representation:
In general, every implicit curve is defined by an equation of the form $f(x, y) = 0,$

for some function f of two variables. Examples of implicit curves include:

A circle: $f(x, y) = x^2 + y^2 - 1,$

An ellipse $f(x, y) = x^2/a^2 + y^2/b^2 - 1.$

A parabola $f(x, y) = y^2 - 4ax$

Conic section $f(x, y) = ax^2 + bxy + cy^2 + dx + ey + f.$

where a, b, c, d, e and f are real constants. The essential disadvantage of an implicit curve is the lack of an easy possibility to calculate single point which is necessary for visualization of an implicit curve.

Definition 1.0.3. Parametric representation:

If the variables x and y are represented as a function of some other independent variable (say t) where t is parameter. It may be a time or an angle. Then such a representation is said to be parametric representation. It is generally denoted as $Q(t) = (x(t), y(t))$.

There are two categories of curves that can be represented parametrically: analytic curves and synthetic curves.

Analytic curves are defined as those that can be described by analytic equations such as lines, circles, and conics. Examples of analytic curves include:

A line: $Q(t) = (at + b, ct + d),$

A circle: $Q(t) = (r \cos(t), r \sin(t)),$

An ellipse: $Q(t) = (a \cos(t), b \sin(t)),$

A parabola: $Q(t) = (at^2, 2at),$

where a, b, c, d, r are constants and t belongs to some interval.

Synthetic curves are the ones that are described by a set of data points (control points) such as Bezier and B-splines curves (will be discussed later). The need for synthetic curves in design arises on two occasions: when a curve is represented by a collection of measured data points and when an existing curve must change to meet new design requirements. Analytic curves are usually not sufficient to meet geometric design requirements of mechanical parts. Synthetic curves provide designers with greater flexibility and control of a curve shape by changing the positions of the control points. Products such as car bodies, ship hulls, airplane fuselage and wings, propeller blades, shoe insoles, and bottles are a few examples that require free-form, or synthetic, curves and surfaces.

1.1 Tangent and normal vectors

A straight line that touches a curve at a point, but if extended does not cross it at that point is called tangent. Formally, it is a line which intersects a differentiable curve at a point where the slope of the curve equals the slope of the line. A normal vector is perpendicular to the tangent. A tangent vector rotated by 90^0, is called normal vector. We find normal vector by multiplying tangent vector with a 90^0 degree rotation matrix

$$\text{Rotation matrix at } 90^0 = \begin{bmatrix} \cos(\frac{\pi}{2}) & -\sin(\frac{\pi}{2}) \\ \sin(\frac{\pi}{2}) & \cos(\frac{\pi}{2}) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

If $Q(t) = (x(t), y(t))$ is parametric curve then tangent $T(t)$ and normal $N(t)$ vectors are

$$T(t) = (x'(t), y'(t)).$$

$$N(t) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -y'(t) \\ x'(t) \end{bmatrix}.$$

The equation of tangent line at $t = t_0$ is

$$L_T(t) = Q(t_0) + T(t_0)t,$$

while the equation of normal line at $t = t_0$ is

$$L_N(t) = Q(t_0) + N(t_0)t.$$

Example 1.1.1. Find the tangent and normal lines of the parametric form of an ellipse at $t_0 = \pi$.

Solution. Since the parametric form of an ellipse is $Q(t) = (a\,\cos(t), b\,\sin(t))$, so

$$T(t) = \frac{d}{dt}Q(t) = Q'(t) = (-a\,\sin(t), b\,\cos(t)),$$

$$N(t) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -a\,\sin(t) \\ b\,\cos(t) \end{bmatrix} = \begin{bmatrix} -b\,\cos(t) \\ -a\,\sin(t) \end{bmatrix}.$$

Or

$$N(t) = \Big(-b\,\cos(t),\ -a\,\sin(t) \Big).$$

Therefore the tangent line

$$\begin{aligned} L_T(t) &= Q(t_\circ) + T(t_\circ)t, \qquad t_\circ = \pi \\ &= (a\,\cos(\pi), b\,\sin(\pi)) + (-a\,\sin(\pi), b\,\cos(\pi))t. \end{aligned}$$

This implies

$$L_T(t) = (-a, 0) + (0, b)t = (-a, bt),$$

and the normal line

$$\begin{aligned} L_N(t) &= Q(t_0) + N(t_\circ)t, \qquad t_0 = \pi \\ &= (a\,\cos(\pi), b\,\sin(\pi)) + (-b\,\cos(\pi), -a\,\sin(\pi))t. \end{aligned}$$

This implies

$$L_N(t) = (-a, 0) + (b, 0)t = (-a + bt, 0).$$

□

Example 1.1.2. Find the tangent and normal lines of the parametric form of circle at $t_0 = \pi$.

Solution. Since the parametric form of circle is $Q(t) = (r\ \cos(t), r\ \sin(t))$, so

$$T(t) = Q'(t) = (-r\ \sin(t), r\ \cos(t)),$$

$$N(t) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -r\ \sin(t) \\ r\ \cos(t) \end{bmatrix} = \begin{bmatrix} -r\ \cos(t) \\ -r\ \sin(t) \end{bmatrix}.$$

Or

$$N(t) = (-r\ \cos(t), -r\ \sin(t)).$$

Therefore the tangent line

$$\begin{aligned} L_T(t) &= Q(t_0) + T(t_0)t, \quad t_0 = \pi \\ &= (r\ \cos(\pi), r\ \sin(\pi)) + (-r\ \sin(\pi), -r\ \cos(\pi))t. \end{aligned}$$

This implies

$$L_T(t) = (-r, 0) + (0, -r)t = (-r, -rt),$$

and the normal line

$$\begin{aligned} L_N(t) &= Q(t_\circ) + N(t_\circ)t \quad t_\circ = \pi \\ &= (r\ \cos(\pi), r\ \sin(\pi)) + (-r\ \cos(\pi), -r\ \sin(\pi))t \\ &= (-r, 0) + (r, 0)t \\ &= (-r + rt, 0). \end{aligned}$$

□

1.2 Parametric and geometric continuity

Mathematically, synthetic and parametric curves represent a curve-fitting problem to construct a smooth curve that passes through given data points. There are two categories of continuities: geometric continuity and parametric continuity.

Definition 1.2.1. ***Parametric continuity*** is achieved by matching the parametric derivatives of adjoining curves at their common boundary.

- Zero-order parametric continuity, or C^0 continuity, yields a position continuous curve. That is curves are joined.
- First-order parametric continuity, or C^1 continuity, imply slope continuous curves. It means first derivatives are equal at joint.
- Second-order parametric continuity, or C^2 continuity, imply curvature continuous curves. It means first and second derivatives are equal at joint.
- nth-order parametric continuity, or C^n continuity, means up to nth derivative are equal at joint.

Definition 1.2.2. In ***geometric continuity***, we only require parametric derivatives of the two sections to be proportional to each other at their common boundary.

- Zero-order geometric continuity, or G^0 continuity, is the same as zero-order parametric continuity. That is, the two curves sections must have the same coordinate position at the boundary point. In other words, curves are joined.
- First order geometric continuity, or G^1 continuity, means that the parametric first derivatives are proportional at the intersection on two successive sections. If we denote the parametric position on the curve as $f(t)$, the direction of the tangent vector $f'(t)$, but not necessarily its magnitude, will be the same for two successive curve sections at their joining point under G^1 continuity. In otherworld, first derivatives are proportional at the join point.
- Second-order geometric continuity, or G^2 continuity, means that both the first and second parametric derivatives of the two curve sections are proportional

at their boundary. Under G^2 continuity, curvatures of two curve sections will match at the joining position. In other words, first and second derivatives are proportional at join point.

- Similarly, nth-order geometric continuity, or G^n continuity, means up to nth derivatives are proportional at join point.

A parametric continuity of order n implies geometric continuity of order n, but not vice-versa. A curve generated with geometric continuity conditions is similar to one generated with parametric continuity, but with slight differences in curve shape. With geometric continuity, the curve is pulled toward the section with the greater tangent vector.
The geometric continuity requires the geometry to be continuous, while parametric continuity requires that the underlying parameterizations be continuous as well. In other words, The parametric continuity indicates smoothness of motion. For examples, while deriving on the car, jumps on the road, is example of parametric continuity that is the parametric continuity studies that how much smooth is the road. While geometric continuity indicates visual smoothness of the curve. For example, movie screen.
Mathematically: Let $f(t)$ be any piecewise function defined over the interval $[a, b]$ and c is any point of $[a, b]$ then $f(t)$ will be

- C^0 continuous if $f(c^-) = f(c^+)$,
- C^1 continuous if $f(c^-) = f(c^+)$ and $f'(c^-) = f'(c^+)$,
- C^2 continuous if $f(c^-) = f(c^+)$, $f'(c^-) = f'(c^+)$ and $f''(c^-) = f''(c^+)$.
- G^0 continuous if $f(c^-) = f(c^+)$,
- G^1 continuous if $f(c^-) = f(c^+)$ and $f'(c^-) = k_1 f'(c^+)$,
- G^2 continuous if $f(c^-) = f(c^+)$, $f'(c^-) = k_1 f'(c^+)$ and $f''(c^-) = k_2 f''(c^+)$,

where k_1, k_2 are constants. Similarly, higher order continuities can be defined.

Example 1.2.1. Consider a camera that moves along an elliptic trajectory, but with an altered parametric representation: At $t = a$ the camera's speed suddenly increased by a factor of three 3. The equation describing the motion is

$$f(t) = \begin{cases} (w\cos(t),\ h\sin(t)), & \text{if } 0 \leqslant t \leqslant a, \\ (w\cos(3t-2a),\ h\sin(3t-2a)), & \text{if } a \leqslant t \leqslant \frac{2(\pi+a)}{3}. \end{cases}$$

Then discuss parametric and geometric continuity upto second order.

Solution. Since f is piecewise function so continuity will be discussed at the joint i.e. at point a. Since

$$\begin{aligned} f(a^-) &= (w\cos(a), h\sin(a)) \\ f(a^+) &= (w\cos(3a-2a), h\sin(3a-2a)) = (w\cos(a), h\sin(a)). \end{aligned}$$

Therefore $f(a^-) = f(a^+)$. Hence f is C^0 and G^0-continuous at point a. Since cos and sin functions are continuous over the entire domain so f is C^0 and G^0-continuous over the entire domain. Since

$$f'(t) = \begin{cases} (-w\sin(t),\ h\cos(t)), & \text{if } 0 \leqslant t \leqslant a, \\ (-3w\sin(3t-2a),\ 3h\cos(3t-2a)), & \text{if } a \leqslant t \leqslant \frac{2(\pi+a)}{3}. \end{cases}$$

Therefore

$$\begin{aligned} f'(a^-) &= (-w\sin(a), h\cos(a)), \\ f'(a^+) &= (-3w\sin(3a-2a), 3h\cos(3a-2a)) = 3(-w\sin(a), h\cos(a)). \end{aligned}$$

So $f'(a^-) \neq f'(a^+)$. Hence f is not C^1 continuous at point a. But as

$$f'(a^+) = 3(-w\sin(a), h\cos(a)) = 3f'(a^-).$$

So f is G^1 continuous at point a as well as on the entire domain. Now since

$$f''(t) = \begin{cases} (-w\cos(t),\ -h\sin(t)), & \text{if } 0 \leqslant t \leqslant a, \\ (-9w\cos(3t-2a),\ -9h\sin(3t-2a)), & \text{if } a \leqslant t \leqslant \frac{2(\pi+a)}{3}. \end{cases}$$

This implies

$$\begin{aligned} f''(a^-) &= (-w\cos(a),\ -h\sin(a)), \\ f''(a^+) &= (-9w\cos(3a-2a),\ -9h\sin(3a-2a)) \\ &= 9(-w\cos(a),\ -h\sin(a)). \end{aligned}$$

As $f''(a^+) = 9f''(a^-)$, then f is G^2 continuous at point a as well as on the entire domain. □

1.3 Polynomials

Explicity the nth degree polynomial can be defined as

$$p(t) = a_n t^n + a_{n-1}t^{n-1} + \cdots + a_2 t^2 + a_1 t + a_0,$$

where $a_n, a_{n-1}, \cdots, a_2, a_1, a_0$ are real coefficients and all the powers are non-negative integers. This polynomial is represented by the linear combination of certain elementary polynomial $1, t^1, t^2, \cdots, t^n$. The order of polynomial is one more than the degree of polynomial. That is if the degree of polynomial is n then its order is $n+1$ (total number of coefficients).

The set of polynomials of degree less than or equal to n forms a vector space and the set of elementary polynomials $\{1, t^1, t^2, \cdots, t^n\}$ forms a basis for this vector space.

A cubic polynomial is the minimum-order polynomial that can guarantee the generation of C^0, C^1 and C^2 curves. The higher-order polynomials are not commonly used in Computer Aided Design (CAD) systems because they tend to oscillate about control points, are computationally inconvenient, and are uneconomical of storing curve representations in the computer.

The parametric polynomial of degree 1 is defined as $p(t) = (at+b, ct+d)$, where a, b, c and d are constants. Of course the shape of this polynomial is a straight line.

The parametric polynomial of degree 2 is defined as $p(t) = (at^2+bt+c, dt^2+et+f)$, where a, b, c, d, e and f are constants and $(a, d \neq 0)$. The shape of this polynomial is parabola. Similarly, one can define higher degree parametric polynomials.

1.3.1 Implicit and rational polynomials of degree 2

The well known implicit 2nd degree polynomial to generate conic section is $f(x,y) = Ax^2 + Bxy + Cy^2 + Dx + Ey + F$, where A, B, C, D, E, F, are real constants. If

- $AC - B^2 > 0$, then $f(x,y)$ represents ellipse,
- $AC - B^2 = 0$, then $f(x,y)$ represents parabola,
- $AC - B^2 < 0$, then $f(x,y)$ represents hyperbola.

Unfortunately, there do not exist 2nd degree parametric polynomial to generate conic section. But luckily, there exists a 2nd degree rational parametric polynomial to represent conic section defined in the form of ratio of two polynomials.

$$p(t) = \frac{p_\circ(1-t)^2 + 2p_1wt(1-t) + p_2t^2}{(1-t)^2 + 2wt(1-t) + t^2}, \quad t \in [0,1],$$

where w is a parameter, $p_0(x_0, y_0), p_1(x_1, y_1), p_2(x_2, y_2)$ are any three points in the plane. These points are known as control points because these points control the shape of curve. The above polynomial can be written as $p(t) = (x(t),\ y(t))$, that is

$$p(t) = \left(\frac{x_0(1-t)^2 + 2x_1wt(1-t) + x_2t^2}{(1-t)^2 + 2wt(1-t) + t^2},\ \frac{y_0(1-t)^2 + 2y_1wt(1-t) + y_2t^2}{(1-t)^2 + 2wt(1-t) + t^2} \right).$$

It has nice properties:

- when $t = 0$, we can get initial point $p(0) = (x_0,\ y_0) = p_0$,
- when $t = 1$, we can get final point $p(1) = (x_2,\ y_2) = p_2$,
- when $0 \leqslant t \leqslant 1$, if
  w <1, then $p(t)$ represents ellipse,
  w=1, then $p(t)$ represents parabola,
  w>1, then $p(t)$ represents hyperbola.

1.4 Bernstein polynomial

Bernstein polynomial was introduced by Sergei Natanovich Bernstein. Sometimes Romanized as Bernshtein (5 March 1880-26 October 1968) was a Russian and Soviet mathematician. Bernstein polynomial of degree n, order $n+1$ is defined by

$$B_{i,n}(t) = \binom{n}{i} t^i (1-t)^{n-i}, \qquad t \in [0,1], \tag{1.1}$$

where $i = 0, 1, \ldots, n$.

- Bernstein polynomials of degree $n = 1$, for $i = 0, 1$, are

$$\begin{aligned} B_{0,1}(t) &= \binom{1}{0} t^0 (1-t)^{1-0} = 1-t, \\ B_{1,1}(t) &= \binom{1}{1} t^1 (1-t)^{1-1} = t. \end{aligned} \tag{1.2}$$

- Bernstein polynomials of degree $n = 2$, for $i = 0, 1, 2$, are

$$\begin{aligned} B_{0,2}(t) &= \binom{2}{0} t^0 (1-t)^{2-0} = (1-t)^2, \\ B_{1,2}(t) &= \binom{2}{1} t^1 (1-t)^{2-1} = 2t(1-t), \\ B_{2,2}(t) &= \binom{2}{2} t^2 (1-t)^{2-2} = t^2. \end{aligned} \tag{1.3}$$

- Bernstein polynomials of degree $n = 3$, for $i = 0, 1, 2, 3$, are

$$\begin{aligned} B_{0,3}(t) &= \binom{3}{0} t^0 (1-t)^{3-0} = (1-t)^3, \\ B_{1,3}(t) &= \binom{3}{1} t^1 (1-t)^{3-1} = 3t(1-t)^2, \\ B_{2,3}(t) &= \binom{3}{2} t^2 (1-t)^{3-2} = 3t^2(1-t), \\ B_{3,3}(t) &= \binom{3}{2} t^3 (1-t)^{3-3} = t^3. \end{aligned} \tag{1.4}$$

1.4.1 Properties of Bernstein polynomials

Some of the important properties of Bernstein polynomials are given below

Property 1.4.1. $B_{0,0}(t) = 1$.

Property 1.4.2. $B_{i,n}(t) = 0$ *when* $i < 0$ *and* $i > n$.

Proof. First consider the case when $i < 0$, that is replace i by $-i$

$$B_{i,n}(t) = \binom{n}{-i} t^{-i}(1-t)^{n+i}.$$

Since $\binom{n}{-i} = 0$, then $B_{i,n}(t) = 0$. Now consider the case when $i > n$. Since $\binom{n}{i} = 0$, for $i > n$, so

$$B_{i,n}(t) = \binom{n}{i} t^{i}(1-t)^{n-i} = 0.$$

This completes the proof. □

Property 1.4.3.

$$B_{i,n}(0) = \begin{cases} 1, & \textit{if} \quad i = 0, \\ 0, & \textit{if} \quad i \neq 0. \end{cases} \tag{1.5}$$

Proof. Consider the case when $i = 0$, that is

$$B_{0,n}(0) = \binom{n}{0} 0^{0}(1-0)^{n-0}.$$

Since, $0^0 = 1$, therefore

$$B_{0,n}(0) = \binom{n}{0} 0^{0}(1)^{n} = 1 \times 1 \times 1 = 1.$$

Now consider the case when $i \neq 0$

$$B_{i,n}(0) = \binom{n}{i} 0^{i}(1)^{n-i}.$$

Since $0^i = 0$ for $i \neq 0$, therefore $B_{i,n}(0) = 0$. □

Property 1.4.4.

$$B_{i,n}(1) = \begin{cases} 1, & \textit{if} \quad i = n, \\ 0, & \textit{if} \quad i = 0, 1, 2, \ldots, n-1. \end{cases} \tag{1.6}$$

Proof. Consider the case when $i = n$, that is

$$B_{n,n}(1) = \binom{n}{n} 1^n (1-1)^{n-n}.$$

Since, $0^0 = 1$, therefore

$$B_{n,n}(1) \;=\; \binom{n}{n} 1^n (0)^0 = 1 \times 1 \times 1 = 1.$$

Now consider the case when $i = 0, 1, 2, \ldots, n-1$.

$$B_{i,n}(1) \;=\; \binom{n}{i} 1^i (0)^{n-i}.$$

Since $n - i \neq 0$, for $i = 0, 1, 2, \ldots, n-1$, then $0^{n-i} = 0$, therefore $B_{i,n}(1) = 0$. □

Property 1.4.5. *Bernstein polynomial of degree n is the linear combination of the Bernstein polynomial of degree $n-1$.*

$$B_{i,n}(t) = (1-t)B_{i,n-1}(t) + tB_{i-1,n-1}(t), \quad i = 1, 2, 3, \cdots, n.$$

Proof. Consider $(1-t)B_{i,n-1}(t) + tB_{i-1,n-1}(t)$

$$\begin{aligned}
&= (1-t)\binom{n-1}{i} t^i (1-t)^{n-1-i} + t\binom{n-1}{i-1} t^{i-1}(1-t)^{n-1-(i-1)} \\
&= (1-t)\binom{n-1}{i} t^i (1-t)^{n-1-i} + t\binom{n-1}{i-1} t^{i-1}(1-t)^{n-i} \\
&= \binom{n-1}{i} t^i (1-t)^{n-i} + \binom{n-1}{i-1} t^i (1-t)^{n-i} \\
&= \left[\binom{n-1}{i} + \binom{n-1}{i-1}\right] t^i (1-t)^{n-i} \\
&= \binom{n}{i} t^i (1-t)^{n-i} \\
&= B_{i,n}(t).
\end{aligned}$$

This completes the proof. $\square$

Property 1.4.6. *Bernstein polynomials form a partition of unity i.e.* $\sum_{i=1}^{n} B_{i,n}(t) = 1$.

Proof. Since $1 = (1-t)+t$, therefore $1^n = [(1-t)+t]^n$. By Binomial theorem

$$\begin{aligned}[(1-t)+t]^n &= \binom{n}{0} t^0(1-t)^{n-0} + \binom{n}{1} t^1(1-t)^{n-1} + \dots \\ &+ \binom{n}{n} t^n(1-t)^{n-n}.\end{aligned}$$

This implies

$$[(1-t)+t]^n = B_{0,n}(t) + B_{1,n}(t) + \cdots + B_{n,n}(t).$$

This again implies

$$1 = 1^n = [(1-t)+t]^n = \sum_{i=0}^{n} B_{i,n}(t). \tag{1.7}$$

This completes the proof. $\square$

Property 1.4.7. *The derivative of Bernstein polynomial of degree* n *is linear combination of* $n-1$ *degree Bernstein polynomial i.e.*

$$\frac{d}{dt} B_{i,n}(t) = n\left[B_{i-1,n-1}(t) - B_{i,n-1}(t)\right], \ \textit{for} \ \ 0 < i < n. \tag{1.8}$$

Proof. Consider $\frac{d}{dt} B_{i,n}(t)$

$$\begin{aligned}&= \frac{d}{dt}\left[\binom{n}{i} t^i(1-t)^{n-i}\right] = \binom{n}{i}\left[it^{i-1}(1-t)^{n-i} - t^i(n-i)(1-t)^{n-i-1}\right] \\ &= \binom{n}{i} it^{i-1}(1-t)^{n-i} - \binom{n}{i} t^i(n-i)(1-t)^{n-i-1}.\end{aligned}$$

This implies

$$\begin{aligned}
&= \frac{n!}{i!(n-i)!} \quad it^{i-1}(1-t)^{n-i} - \frac{n!}{i!(n-i)!} \quad t^i(n-i)(1-t)^{n-i-1} \\
&= \frac{n(n-1)!}{i(i-1)!(n-1-(i-1))!} \quad it^{i-1}(1-t)^{n-i} \\
&\quad -\frac{n(n-1)!}{i!(n-i)(n-i-1)!} \quad t^i(n-i)(1-t)^{n-i-1} \\
&= n\binom{n-1}{i-1}t^{i-1}(1-t)^{n-1} - n\binom{n-1}{i}t^i(1-t)^{n-i-1}.
\end{aligned}$$

this again implies

$$\frac{d}{dt}B_{i,n}(t) \quad = \quad n\left[B_{i-1,n-1}(t) - B_{i,n-1}(t)\right].$$

This completes the proof. □

Property 1.4.8. *Bernstein polynomials are all non-negative, for* $0 \leqslant t \leqslant 1$.

Proof. By definition

$$B_{i,n} = \binom{n}{i}t^i(1-t)^{n-i}.$$

Since $t \geqslant 0$ and $(1-t) \geqslant 0$ for $0 \leqslant t \leqslant 1$, therefore $B_{i,n} \geqslant 0$ □

1.4.2 Degree raising in Bernstein polynomial

Any of the lower degree Bernstein polynomial of degree less than n can be expressed as a linear combination of Bernstein polynomial of degree n.

Theorem 1.4.9. *Prove that any Bernstein polynomial of degree* $n-1$ *can be written as a linear combination of degree* n.

Proof. Consider the Bernstein polynomial of degree n

$$\begin{aligned}
B_{i,n}(t) &= 1B_{i,n}(t) = (1-t+t)B_{i,n}(t) \\
&= (1-t)B_{i,n}(t) + tB_{i,n}(t).
\end{aligned}$$

Consider

$$\begin{aligned} tB_{i,n}(t) &= t\binom{n}{i}t^i(1-t)^{n-i} = \binom{n}{i}t^{i+1}(1-t)^{(n+1)-(i+1)} \\ &= \frac{\binom{n}{i}}{\binom{n+1}{i+1}}\binom{n+1}{i+1}t^{i+1}(1-t)^{(n+1)-(i+1)}. \end{aligned}$$

This implies

$$tB_{i,n}(t) = \frac{\frac{n!}{i!(n-i)!}}{\frac{(n+1)!}{(i+1)!(n+1)-(i+1)!}}B_{i+1,n+1}(t) = \frac{\frac{n!}{i!(n-i)!}}{\frac{(n+1)!}{(i+1)!(n-i)!}}B_{i+1,n+1}(t).$$

Again implies

$$tB_{i,n}(t) = \frac{n!}{i!(n-i)!}\frac{(i+1)!(n-i)!}{(n+1)!}B_{i+1,n+1}(t).$$

Or

$$tB_{i,n}(t) = \frac{n!}{i!(n-i)!}\frac{(i+1)i!(n-i)!}{(n+1)n!}B_{i+1,n+1}(t).$$

Further implies

$$tB_{i,n}(t) = \left(\frac{i+1}{n+1}\right)B_{i+1,n+1}(t).$$

Now consider

$$\begin{aligned} (1-t)B_{i,n}(t) &= (1-t)\binom{n}{i}t^i(1-t)^{n-i} = \binom{n}{i}t^i(1-t)^{n+1-i} \\ &= \frac{\binom{n}{i}}{\binom{n+1}{i}}\binom{n+1}{i}t^i(1-t)^{n+1-i} = \frac{\binom{n}{i}}{\binom{n+1}{i}}B_{i,n+1}(t). \end{aligned}$$

This implies

$$(1-t)B_{i,n}(t) = \frac{\frac{n!}{i!(n-i)!}}{\frac{(n+1)!}{i!(n+1-i)!}}B_{i,n+1} = \frac{n!}{i!(n-i)!}\frac{i!(n+1-i)!}{(n+1)!}B_{i,n+1}.$$

Again implies

$$(1-t)B_{i,n}(t) = \frac{n!}{i!(n-i)!}\frac{i!(n+1-i)(n-i)!}{(n+1)n!}B_{i,n+1} = \frac{n+1-i}{n+1}B_{i,n+1}.$$

Further implies

$$(1-t)B_{i,n}(t) = \left(1-\frac{i}{n+1}\right)B_{i,n+1}.$$

Now add both terms

$$\begin{aligned} B_{i,n}(t) &= (1-t)B_{i,n}(t) + tB_{i,n}(t). \\ &= \left(1-\frac{i}{n+1}\right)B_{i,n+1} + \left(\frac{i+1}{n+1}\right)B_{i+1,n+1}(t). \end{aligned}$$

Replace n by $n-1$

$$B_{i,n-1}(t) = \left(1-\frac{i}{n}\right)B_{i,n} + \left(\frac{i+1}{n}\right)B_{i+1,n}(t).$$

This completes the proof □

1.4.3 Converting from Bernstein basis to power basis

We know that $1, t, t^2, t^3, \cdots, t^n$ are called power basis or monomials.

Theorem 1.4.10. *Prove that every Bernstein polynomial of degree n can be written in terms of power basis $1, t, t^2, t^3, \cdots, t^n$.*

Proof. Consider an arbitrary Bernstein polynomial

$$B_{k,n}(t) = \binom{n}{k}t^k(1-t)^{n-k}.$$

By Binomial theorem

$$\begin{aligned} (1-t)^{n-k} &= \binom{n-k}{0}(1)^{n-k}(-t)^0 + \binom{n-k}{1}(1)^{n-k-1}(-t)^1 \\ &\quad + \binom{n-k}{2}(1)^{n-k-2}(-t)^2 + \cdots\cdots + \binom{n-k}{n-k}(1)^0(-t)^{n-k}. \end{aligned}$$

This implies

$$(1-t)^{n-k} \quad = \quad \sum_{i=0}^{n-k} \binom{n-k}{i} (-1)^i (t)^i.$$

So

$$B_{k,n}(t) \quad = \quad \binom{n}{k} t^k \left\{ \sum_{i=0}^{n-k} \binom{n-k}{i} (-1)^i (t)^i \right\}.$$

This implies

$$B_{k,n}(t) \quad = \quad \sum_{i=0}^{i=n-k} \binom{n}{k} \binom{n-k}{i} (-1)^i (t)^{k+i}.$$

Now replace $k+i$ by i that is replace i by $i-k$, in other word put $i = i-k$

$$B_{k,n}(t) \quad = \quad \sum_{i-k=0}^{i-k=n-k} \binom{n}{k} \binom{n-k}{i-k} (-1)^{i-k} (t)^{k+(i-k)}.$$

Further implies

$$B_{k,n}(t) \quad = \quad \sum_{i=k}^{i=n} \binom{n}{k} \binom{n-k}{i-k} (-1)^{i-k} (t)^i.$$

Since $B_{k,n}(t)$ is arbitrary Bernstein polynomial of degree n, therefore every Bernstein polynomial of degree n can be written as linear combination of monomials. This completes the proof. □

Example 1.4.1. Let $P^3(t)$ be the set of polynomials of degree ≤ 3 then prove that the set of power basis $\{1, t, t^2, t^3\}$ is the basis of $P^3(t)$.

Solution. Let $S = \{1, t, t^2, t^3\}$, to show that S is the basis of $P^3(t)$, we have to prove that

- S is linearly independent,
- S generates (i.e. span) $P(t)$. This means every polynomial of degree less than or equal to 3 can be expressed as the linear combination of $1, t, t^2, t^3$.

First we prove that S is linearly independent. For this consider the linear combination of $1, t, t^2, t^3$ and put it as 0 i.e.

$$a_0 t^0 + a_1 t^1 + a_2 t^2 + a_3 t^3 = 0,$$

where a_0, a_1, a_2, a_3 are scalars. This can be written as

$$a_0 t^0 + a_1 t^1 + a_2 t^2 + a_3 t^3 = 0t^0 + 0t^1 + 0t^2 + 0t^3.$$

Comparing the coefficients of t and constant term, we get $a_0 = a_1 = a_2 = a_3 = 0$. Since all the scalars are zero so S is linearly independent.
Now prove that S generates $P^3(t)$. Let us take an arbitrary element of $P^3(t)$, i.e.

$$at^0 + bt^1 + ct^2 + dt^3 \in P^3(t),$$

where a, b, c and d are known constants. This element is equal to the linear combination of the elements of S, so S generates $P^3(t)$. In other words, the power basis $\{1, t, t^2, t^3\}$ form a space of polynomials of degree less than or equal to 3. This completes the proof. □

Similarly, we can prove that the power basis $\{1, t, t^2, t^3, \ldots t^n\}$ form a space of polynomials of degree less than or equal to n.

Example 1.4.2. Prove that the set of Bernstein polynomials S={$B_{0,3}(t)$, $B_{1,3}(t)$, $B_{2,3}(t)$, $B_{3,3}(t)$} of degree three is the basis of a space of polynomials $P^3(t)$ of degree less than or equal to 3.

Solution. First prove that S is linearly independent, for this take linear combination of elements of S and set it zero

$$a_0 B_{0,3}(t) + a_1 B_{1,3}(t) + a_2 B_{2,3}(t) + a_3 B_{3,3}(t) = 0.$$

By (1.4), this implies

$$a_0(1-t)^3 + a_1 3t(1-t)^2 + a_2 3t^2(1-t) + a_3 t^3 = 0.$$

This implies

$$a_0(1 - t^3 - 3t + 3t^2) + a_1 3t(1 + t^2 - 2t) + a_2(3t^2 - 3t^3) + a_3 t^3 = 0.$$

This further implies

$$\begin{aligned}
&(-a_0 + 3a_1 - 3a_2 + a_3)t^3 + (3a_0 - 6a_1 + 3a_2)t^2 + (-3a_0 + 3a_1)t + a_0 \\
&= 0t^3 + 0t^2 + 0t + 0.
\end{aligned}$$

Comparing the coefficients of t^3, t^2, t and constant term

$$\begin{aligned}
-a_0 + 3a_1 - 3a_2 + a_3 &= 0 \\
3a_0 - 6a_1 + 3a_2 &= 0 \\
-3a_0 + 3a_1 &= 0 \\
a_0 &= 0
\end{aligned}$$

Solving above system of equations by backwards substitution, we get $a_0 = a_1 = a_2 = a_3 = 0$. Since all the scalars are zero therefore S is linearly independent. Now we prove that S generates $P^3(t)$. That is every element of $P^3(t)$ can be written as the linear combination of the elements of S. Consider the element of $P^3(t)$ i.e.

$$at^3 + bt^2 + ct + d \; \in \; P^3(t),$$

where a, b, c and d are known constants. Suppose

$$at^3 + bt^2 + ct + d \;\; = \;\; a_0 B_{0,3}(t) + a_1 B_{1,3}(t) + a_2 B_{2,3}(t) + a_3 B_{3,3}(t).$$

Now find the unknown constants a_0, a_1, a_2 and a_3. By (1.4)

$$\begin{aligned}
& at^3 + bt^2 + ct + d \\
= \;& a_0(1 - t)^3 + a_1 3t(1 - t)^2 + a_2 3t^2(1 - t) + a_3 t^3 \\
= \;& a_0(1 - t^3 - 3t + 3t^2) + a_1 3t(1 + t^2 - 2t) + a_2(3t^2 - 3t^3) + a_3 t^3.
\end{aligned}$$

$$\begin{aligned}
&= a_0(1-t^3-3t+3t^2)+a_1(3t+3t^3-6t^2)+a_2(3t^2-3t^3)+a_3t^3 \\
&= (-a_0+3a_1-3a_2+a_3)t^3+(3a_0-6a_1+3a_2)t^2+(-3a_0+3a_1)t+a_0.
\end{aligned}$$

Comparing the coefficients of t^3, t^2, t and constant term

$$\begin{aligned}
-a_0+3a_1-3a_2+a_3 &= a \\
3a_0-6a_1+3a_2 &= b \\
-3a_0+3a_1 &= c \\
a_0 &= d.
\end{aligned}$$

Since a, b, c and d are known constants, therefore $a_0 = d$ is known. By backwards substitutions, we can find other unknowns. Since $-3a_0+3a_1 = c$, so by substituting $a_0 = d$, we get $-3d+3a_1 = c$, this implies $a_1 = (c+3d)/3$.
Now since

$$3a_0-6a_1+3a_2 = b,$$

so by substituting the values of $a_0 = d$ and $a_1 = (c+3d)/3$,

$$3d-6\{(c+3d)/3\}+3a_2 = b.$$

This implies

$$-3d-2c+3a_2 = b.$$

Again implies

$$a_2 = (b+3d+2c)/3.$$

Now since

$$-a_0+3a_1-3a_2+a_3 = a.$$

By substituting the values of $a_0 = d$, $a_1 = (c+3d)/3$ and $a_2 = (b+2c+3d)/3$, we get

$$-d + 3\{(c+3d)/3\} - 3\{(b+3d+2c)/3\} + a_3 \quad = \quad a.$$

This implies

$$-d - c - b + a_3 \quad = \quad a.$$

Again implies

$$a_3 \quad = \quad a + b + c + d.$$

Hence by substituting the values of a_0, a_1, a_2 and a_3, we get

$$\begin{aligned} at^3 + bt^2 + ct + d &= a_0 B_{0,3}(t) + a_1 B_{1,3}(t) + a_2 B_{2,3}(t) + a_3 B_{3,3}(t) \\ &= \{d_0\} B_{0,3}(t) + \{(c+3d)/3\} B_{1,3}(t) \\ &\quad + \{(b+3d+2c)/3\} B_{2,3}(t) + \{a+b+c+d\} B_{3,3}(t). \end{aligned}$$

This means every elements of $P^3(t)$ is the linear combination of Bernstein polynomials of degree 3. So the set S generates $P^3(t)$. Since S is also linearly independent. Hence the set of Bernstein polynomials of degree 3 is the basis of a space of polynomials of degree less than or equal to 3. $\square$

1.5 Pascal triangle

When the Binomial coefficient $\binom{n}{i}$ in (1.1) is evaluated for different values of i and n then it shows the pattern of numbers shown in Table 1.1. This pattern of number is known as Pascal's triangle. In western countries they are named after a 17th century French mathematician Blaise Pascal, even though they had been described in China as early as 1303 in "Previous Mirror of the Four Elements" by Chinese mathematician Chu Shih-Chieh.

The pattern represents the coefficients found in binomial expansions. For example, the expansion of $(x+a)^n$ for different values of n is which reveals Pascal's triangle as coefficients of the polynomial terms (see (1.9) and Table 1.1). Alternate method to build the triangle is: start with "1" at the top, then continue placing numbers below it in a triangular pattern. Each number is the sum of the two numbers directly above it. It is shown in Table 1.2.

The powers of t and $(1-t)$ in (1.1) appear as shown in Table 1.3 for different values of i and n. When the two sets of results (i.e. in Table 1.1 and 1.3) are combined, we get the complete Bernstein polynomial terms shown in Table 1.4.

Table 1.1: Pascal's triangle

	i				
n	0	1	2	3	4
0	1				
1	1	1			
2	1	2	1		
3	1	3	3	1	
4	1	4	6	4	1

$$
\begin{aligned}
(x+a)^0 &= 1 \\
(x+a)^1 &= 1x + 1a \\
(x+a)^2 &= 1x^2 + 2ax + 1a^2 \\
(x+a)^3 &= 1x^3 + 3ax^2 + 3a^2x + 1a^3 \\
(x+a)^4 &= 1x^4 + 4ax^3 + 6a^2x^2 + 4a^3x + 1a^4. \qquad (1.9)
\end{aligned}
$$

Table 1.2: Pascal's triangle

n/i	0		1		2		3		4	
0						1				
1					1		1			
2				1		2		1		
3			1		3		3		1	
4		1		4		6		4		1

Table 1.3: Expansion of the terms t and $(1-t)$

	i				
n	0	1	2	3	4
1	t	$(1-t)$			
2	t^2	$t(1-t)$	$(1-t)^2$		
3	t^3	$t^2(1-t)$	$t(1-t)^2$	$(1-t)^3$	
4	t^4	$t^3(1-t)$	$t^2(1-t)^2$	$t(1-t)^3$	$(1-t)^4$

Table 1.4: The Bernstein polynomial terms

	i				
n	0	1	2	3	4
1	$1t$	$1(1-t)$			
2	$1t^2$	$2t(1-t)$	$1(1-t)^2$		
3	$1t^3$	$3t^2(1-t)$	$3t(1-t)^2$	$1(1-t)^3$	
4	$1t^4$	$4t^3(1-t)$	$6t^2(1-t)^2$	$4t(1-t)^3$	$1(1-t)^4$

1.6 Be′zier curve

Be′zier curve was developed by Paul de Casteljau in 1959 and independently by Pierre Be′zier around 1962. Given the set of control points $\{p_0, p_1, p_2 \cdots, p_n\}$, we can define Be′zeir curve of degree n as

$$P(t) = \sum_{i=0}^{n} p_i B_{i,n}(t), \quad t \in [0,1], \tag{1.10}$$

where $B_{i,n}(t) = \binom{n}{i} t^i (1-t)^{n-i}$ is a Bernstein polynomial of degree n. If $p_i = (x_i, y_i)$ then (1.10) can be written as

$$P(t) = (x(t), y(t)) \;=\; \left(\sum_{i=0}^{n} x_i B_{i,n}(t), \sum_{i=0}^{n} y_i B_{i,n}(t) \right). \tag{1.11}$$

Example 1.6.1. Find the Be′zeir curve which has control points $(2,2)$, $(1,1.5)$, $(3.5,0)$, $(4,1)$.

Solution. Let

$$x(t) \;=\; \sum_{i=0}^{3} x_i B_{i,3}(t) = x_0 B_{0,3}(t) + x_1 B_{1,3}(t) + x_2 B_{2,3}(t) + x_3 B_{3,3}(t).$$

Then by substituting the values of Bernstein polynomials $B_{i,3}(t)$, for $i = 0, 1, 2, 3$

given in (1.4) and using the data points $(x_i, y_i) = (2,2)$, $(1,1.5)$, $(3.5,0)$, $(4,1)$, we get

$$\begin{aligned} x(t) &= 2(1-t)^3 + 1(3t)(1-t)^2 + 3.5(3t^2(1-t)) + 4t^3 \\ &= 2(1-t^3-3t+3t^2) + 3t(1+t^2-2t) + 10.5t^2 - 10.5t^3 + 4t^3 \\ &= 2 - 2t^3 - 6t + 6t^2 + 3t + 3t^3 - 6t^2 - 10.5t^3 + 10.5t^2 + 4t^3 \\ &= 2 - 3t + 10.5t^2 - 5.5t^3. \end{aligned}$$

Again let

$$y(t) = \sum_{i=0}^{3} y_i B_{i,3}(t) = y_0 B_{0,3}(t) + y_1 B_{1,3}(t) + y_2 B_{2,3}(t) + y_3 B_{3,3}(t).$$

Then by substituting the values of Bernstein polynomials, we get

$$\begin{aligned} y(t) &= 2(1-t)^3 + 1.5(3t)(1-t)^2 + 0(3t^2(1-t)) + 1t^3 \\ &= 2(1-t^3-3t+3t^2) + 1.5(3t(1+t^2-2t)) + t^3 \\ &= 2 - 2t^3 - 6t + 6t^2 + 4.5t + 4.5t^3 - 9t^2 + t^3 \\ &= 2 - 1.5t - 3t^2 + 3.5t^3. \end{aligned}$$

Therefore by (1.11), we get the Be'zeir curve

$$P(t) = (x(t),\ y(t)) = \left(2 - 3t + 10.5t^2 - 5.5t^3,\ 2 - 1.5t - 3t^2 + 3.5t^3\right).$$

□

Example 1.6.2. Find the Be'zeir curves $P(t)$ and $Q(t)$ of degree 3 which have set of control points $\{(0,3),(1,5),(2,1),(3,3)\}$ and $\{(3,3),(4,5),(5,1),(6,3)\}$ respectively.

Solution. First we find Be'zeir curve $P(t) = (x(t), y(t))$ of degree 3, having the set of control points $(x_i, y_i) = \{(0,3),(1,5),(2,1),(3,3)\}$. Since

$$x(t) = \sum_{i=0}^{3} x_i B_{i,n}(t) = x_0 B_{0,3}(t) + x_1 B_{1,3}(t) + x_2 B_{2,3}(t) + x_3 B_{3,3}(t).$$

Then by using (1.4), and $(x_i, y_i) = \{(0,3),(1,5),(2,1),(3,3)\}$

$$\begin{aligned} x(t) &= (0)(1-t)^3 + (1)(3t)(1-t)^2 + (2)(3t^2(1-t)) + 3t^3 \\ &= 3t + 3t^3 - 6t^2 + 6t^2 - 6t^3 + 3t^3 \\ &= 3t \end{aligned}$$

Now since

$$y(t) = \sum_{i=0}^{3} y_i B_{i,3}(t) = y_0 B_{0,3}(t) + y_1 B_{1,3}(t) + y_2 B_{2,3}(t) + y_3 B_{3,3}(t).$$

Then by using (1.4), and $(x_i, y_i) = \{(0,3),(1,5),(2,1),(3,3)\}$, we get

$$\begin{aligned} y(t) &= 3(1-t)^3 + 5(3t)(1-t)^2 + (1)3t^2(1-t) + 3t^3 \\ &= 3(1-t^3-3t+3t^2) + 15(3t(1+t^2-2t)) + +3t^2 - 3t^3 + 3t^3 \\ &= 12t^3 - 18t^2 + 6t + 3. \end{aligned}$$

Thus the Be'zeir curve $P(t)$ of degree 3, having the set of control points $(x_i, y_i) = \{(0,3),(1,5),(2,1),(3,3)\}$ is

$$P(t) = (x(t), y(t)) = \left(3t, 12t^3 - 18t^2 + 6t + 3\right). \tag{1.12}$$

Now we find Be'zeir curve $Q(t) = (x(t), y(t))$ of degree 3, having the set of control points $(x_i, y_i) = \{(3,3),(4,5),(5,1),(6,3)\}$. Since

$$x(t) = \sum_{i=0}^{3} x_i B_{i,n}(t) = x_0 B_{0,3}(t) + x_1 B_{1,3}(t) + x_2 B_{2,3}(t) + x_3 B_{3,3}(t).$$

Then by using (1.4), and $(x_i, y_i) = \{(3,3),(4,5),(5,1),(6,3)\}$

$$\begin{aligned} x(t) &= (3)(1-t)^3 + (4)(3t)(1-t)^2 + (5)(3t^2(1-t)) + 6t^3 \\ &= 3 - 3t^3 + -9t + 9t^2 + 12t + 12t^3 - 24t^2 + 15t^2 - 15t^3 + 6t^3 \\ &= 3 + 3t. \end{aligned}$$

Again since

$$y(t) = \sum_{i=0}^{3} y_i B_{i,3}(t) = y_0 B_{0,3}(t) + y_1 B_{1,3}(t) + y_2 B_{2,3}(t) + y_3 B_{3,3}(t).$$

Then by using (1.4), and $(x_i, y_i) = \{(3,3),(4,5),(5,1),(6,3)\}$

$$\begin{aligned} y(t) &= 3(1-t)^3 + 5(3t)(1-t)^2 + 1(3t^2(1-t)) + 3t^3 \\ &= 3(1-t^3-3t+3t^2) + 15(3t(1+t^2-2t)) + +3t^2 - 3t^3 + 3t^3 \\ &= 12t^3 - 18t^2 + 6t + 3. \end{aligned}$$

Thus the Be'zeir curves $Q(t)$ of degree 3, having the set of control points $(x_i, y_i) = \{(3,3),(4,5),(5,1),(6,3)\}$ is

$$Q(t) = (x(t), y(t)) = \left(3 + 3t, 12t^3 - 18t^2 + 6t + 3\right). \tag{1.13}$$

□

1.6.1 Properties of Be'zeir curve

Property 1.6.1. *Be'zeir curve is continuous and has continuous derivatives on the interval* $[0,1]$.

Proof. As we know that Be'zeir curve is

$$P(t) = \sum_{i=0}^{n} p_i B_{i,n}(t), \quad t \in [0,1],$$

where $B_{i,n}(t)$ are Bernstein polynomials. We know that every polynomial is continuous and has continuous derivatives of all order. It follows that Bezier curve is also continuous and all its derivative exists and are continuous. This completes the proof. □

Property 1.6.2. ***Endpoint Interpolation property:*** *Be'zeir curve $P(t)$ interpolates / passes through initial p_0 and final p_n points of its control polygon.*

Proof. Consider Be′zeir curve with control points $\{p_0, p_1, p_2, \cdots, p_n\}$

$$P(t) = \sum_{i=0}^{n} p_i B_{i,n}(t).$$

We will prove that $P(0) = p_0$ and $P(1) = p_n$. Now, for $t = 0$

$$\begin{aligned} P(0) &= \sum_{i=0}^{n} p_i B_{i,n}(0) = p_0 B_{0,n}(0) + p_1 B_{1,n}(0) + p_2 B_{2,n}(0) + \cdots \\ &\quad + p_n B_{n,n}(0). \end{aligned}$$

By using (1.5)

$$P(0) = p_0(1) + p_1(0) + p_2(0) + \cdots + p_n(0) = p_0.$$

For $t = 1$

$$\begin{aligned} P(1) &= \sum_{i=0}^{n} p_i B_{i,n}(1) = p_0 B_{0,n}(1) + p_1 B_{1,n}(1) + p_2 B_{2,n}(1) + \cdots \\ &\quad + p_n B_{n,n}(1). \end{aligned}$$

By using (1.6)

$$P(1) = p_0(0) + p_1(0) + p_2(0) + \cdots + p_n(1) = p_n.$$

Hence Be′zeir curve always passes through initial and final pints. This completes the proof. □

Property 1.6.3. *The tangent to a Be′zeir curve at the end points are parallel to the lines through end points and adjacent control points.*

Proof. If $P(t)$ is a Be′zeir curve and $P'(t)$ is it derivative then we will proof that
Tangent $P'(0)$ at p_0 is parallel to the line passes through p_0 and p_1.
Tangent $P'(1)$ at p_n is parallel to the line passes through p_{n-1} and p_n.

In other word, we will prove that

$$\begin{aligned} P'(0) &= n(p_1 - p_0), \\ P'(1) &= n(p_n - p_{n-1}). \end{aligned} \tag{1.14}$$

Taking derivative of (1.10)

$$P'(t) = \sum_{i=0}^{n} p_i \frac{d}{dt}(B_{i,n}(t)).$$

By (1.8), we have

$$P'(t) = n \sum_{i=0}^{n} p_i \left[B_{i-1,n-1}(t) - B_{i,n-1}(t)\right]. \tag{1.15}$$

For $t = 0$, we get

$$\begin{aligned} P'(0) &= n \sum_{i=0}^{n} p_i \left[B_{i-1,n-1}(0) - B_{i,n-1}(0)\right] \\ &= n \left[p_0 \left\{B_{0-1,n-1}(0) - B_{0,n-1}(0)\right\} + p_1 \left\{B_{0,n-1}(0) - B_{1,n-1}(0)\right\}\right. \\ &\quad \left. + p_2 \left\{B_{1,n-1}(0) - B_{2,n-1}(0)\right\} + \cdots + p_n \left\{B_{n-1,n-1}(0) - B_{n,n-1}(0)\right\}\right]. \end{aligned}$$

By using (1.5), we get

$$\begin{aligned} P'(0) &= n \left[p_0 \{0 - 1\} + p_1 \{1 - 0\} + p_2 \{0 - 0\} + \cdots + p_n \{0 - 0\}\right] \\ &= n(p_1 - p_0). \end{aligned}$$

Now for $t = 1$

$$\begin{aligned} P'(1) &= n \sum_{i=0}^{n} p_i \left[B_{i-1,n-1}(1) - B_{i,n-1}(1)\right] \\ &= n \left[p_0 \left\{B_{0-1,n-1}(1) - B_{0,n-1}(1)\right\} + p_1 \left\{B_{0,n-1}(1) - B_{1,n-1}(1)\right\}\right. \\ &\quad \left. + p_2 \left\{B_{1,n-1}(1) - B_{2,n-1}(1)\right\} + \cdots + p_n \left\{B_{n-1,n-1}(1) - B_{n,n-1}(1)\right\}\right]. \end{aligned}$$

By using (1.6), we get

$$\begin{aligned} P'(1) &= np_0\{0-0\}+p_1\{0-0\}+p_2\{0-0\}+\cdots \\ &\quad +p_{n-1}\{0-1\}+p_n\{1-0\}=n(p_n-p_{n-1}). \end{aligned}$$

This completes the proof. □

Property 1.6.4. *The curvature at the end points of control polygon of Be'zeir curve are parallel to $p_2-2p_1+p_0$ and $p_n-2p_{n-1}+p_{n-2}$.*

Proof. If $P(t)$ is a Be'zeir curve and $P''(t)$ is its 2nd derivative then, we will proof that

$$\begin{aligned} P''(0) &= n(n-1)(p_2-2p_1+p_0), \\ P''(1) &= n(n-1)(p_n-2p_{n-1}+p_{n-2}). \end{aligned} \tag{1.16}$$

By (1.15), we have

$$P'(t)=n\sum_{i=0}^{n}p_i\{B_{i-1,n-1}(t)-B_{i,n-1}(t)\}.$$

Now by taking derivative, we get

$$P''(t)=n\sum_{i=0}^{n}p_i\left\{\frac{d}{dt}B_{i-1,n-1}(t)-\frac{d}{dt}B_{i,n-1}(t)\right\}.$$

By using (1.8)

$$\begin{aligned} P''(t) &= n(n-1)\sum_{i=0}^{n}p_i\{[B_{i-2,n-2}(t)-B_{i-1,n-2}(t)] \\ &\quad -[B_{i-1,n-2}(t)-B_{i,n-2}(t)]\} \\ &= n(n-1)\sum_{i=0}^{n}p_i\left[B_{i-2,n-2}(t)-2B_{i-1,n-2}(t)+B_{i,n-2}(t)\right]. \end{aligned}$$

For $t=0$

$$P''(0) = n(n-1)\sum_{i=0}^{n} p_i \left[B_{i-2,n-2}(0) - 2B_{i-1,n-2}(0) + B_{i,n-2}(0)\right].$$

Expanding

$$\begin{aligned} P''(0) = {} & n(n-1)\{p_0\left[B_{-2,n-2}(0) - 2B_{-1,n-2}(0) + B_{0,n-2}(0)\right] \\ & + p_1\left[B_{-1,n-2}(0) - 2B_{0,n-2}(0) + B_{1,n-2}(0)\right] \\ & + p_2\left[B_{0,n-2}(0) - 2B_{1,n-2}(0) + B_{2,n-2}(0)\right] \\ & + \cdots\cdots \\ & + p_n\left[B_{n-2,n-2}(0) - 2B_{n-1,n-2}(0) + B_{n,n-2}(0)\right]\}. \end{aligned}$$

By using (1.5)

$$\begin{aligned} P''(0) &= n(n-1)\{p_0[0-0+1] + p_1[0-2+0] + p_2[1-0+0] \\ &\quad + p_3[0-0+0] + \cdots\cdots + p_n[0-0+0]\} \\ &= n(n-1)(p_2 - 2p_1 + p_0). \end{aligned}$$

Now for $t = 1$

$$P''(1) = n(n-1)\sum_{i=0}^{n} p_i \left[B_{i-2,n-2}(1) - 2B_{i-1,n-2}(1) + B_{i,n-2}(1)\right].$$

Expanding

$$\begin{aligned} P''(1) = {} & n(n-1)\{p_0\left[B_{-2,n-2}(1) - 2B_{-1,n-2}(1) + B_{0,n-2}(1)\right] \\ & + p_1\left[B_{-1,n-2}(1) - 2B_{0,n-2}(1) + B_{1,n-2}(1)\right] \\ & + p_2\left[B_{0,n-2}(1) - 2B_{1,n-2}(1) + B_{2,n-2}(1)\right] \\ & + p_3\left[B_{1,n-2}(1) - 2B_{2,n-2}(1) + B_{3,n-2}(1)\right] + \cdots \\ & + p_n\left[B_{n-2,n-2}(1) - 2B_{n-1,n-2}(1) + B_{n,n-2}(1)\right]\}. \end{aligned}$$

Again by (1.6)

$$\begin{aligned}
P''(1) &= n(n-1)\{p_0[0-0+0]+p_1[0-0+0]+p_2[0-0+0] \\
&\quad +p_3[0-0+0]+\cdots\cdots+p_{n-2}[0-0+1]+p_{n-1}[0-2+0] \\
&\quad +p_n[1-0+0]\} \\
&= n(n-1)(p_n-2p_{n-1}+p_{n-2}).
\end{aligned}$$

This completes the proof. □

Example 1.6.3. Consider the composite Be'zier curve made by Be'zier curves $P(t)$ and $Q(t)$ of degree 3 with points sets: $\{(0,3),(1,5),(2,1),(3,3)\}$ and $\{(3,3),(4,5),$ $(5,1),(6,3)\}$ respectively. Then discuss the parametric and geometric continuities.

Solution. Since by (1.12) and (1.13), we have

$$\begin{aligned}
P(t) &= \left(3t, 12t^3-18t^2+6t+3\right), \\
Q(t) &= \left(3+3t, 12t^3-18t^2+6t+3\right),
\end{aligned}$$

This implies

$$\begin{aligned}
P'(t) &= \left(3, 36t^2-36t^1+6\right), \qquad P''(t) = \left(0, 72t^1-36\right), \\
Q'(t) &= \left(3, 36t^2-36t^1+6\right), \qquad Q''(t) = \left(0, 72t^1-36\right),
\end{aligned}$$

$$\begin{aligned}
P'''(t) &= (0,72), \qquad P^{iv}(t) = (0,0), \\
Q'''(t) &= (0,72), \qquad Q^{iv}(t) = (0,0).
\end{aligned}$$

We see that $P(1)=Q(0)=(3,3)$, i.e. last point of $P(t)$ and initial point of $Q(t)$ are same, so $P(t)$ and $Q(t)$ are join together. In other word, both pieces are joined by C^0-continuity as well as G^0-continuity. Since $P(t)$ and $Q(t)$ are polynomials and polynomials are continuous, therefore composite Be'zier curve is C^0 as well as G^0-continuous over the entire domain.

Now since $P(1)=Q(0)=(3,3)$ and $P'(1)=Q'(0)=(3,6)$, this means $P(t)$ and $Q(t)$ are join together by C^1-continuity as well as G^1-continuity. Since $P(t)$ and

$Q(t)$ are polynomials and polynomials are continuous, therefore composite Be'zier curve is C^1 as well as G^1-continuous over the entire domain.

Now since $P(1) = Q(0) = (3,3)$ and $P'(1) = Q'(0) = (3,6)$ and $P''(1) = (0,36) \neq Q''(0)$, this means $P(t)$ and $Q(t)$ are not joined together by C^2-continuity. But $P(1) = Q(0) = (3,3)$ and $P'(1) = Q'(0) = (3,6)$ and $Q''(0) = (0,-36) = -1(0,36) = -1P''(1)$, this means $P(t)$ and $Q(t)$ are joined together by G^2 -continuity. Therefore composite Be'zier curve is G^2 continuous over the entire domain but not C^2 continuous.

Since $P(1) = Q(0) = (3,3)$, $P'(1) = Q'(0) = (3,6)$, $Q''(0) = -1P''(1)$, and $P'''(1) = Q'''(0) = (0,72)$, therefore composite Be'zier curve is G^3-continuous over the entire domain. Similarly, we see that composite Be'zier curve is G^∞-continuous over the entire domain. This completes the proof. □

Alternate method to check C^0, C^1 and C^2-continuities:

Solution. From above example, $\{p_0, p_1, p_2, p_3\}$=$\{(0,3),(1,5),(2,1),(3,3)\}$ and $\{q_0, q_1, q_2, q_3\}$ = $\{(3,3),(4,5),(5,1),(6,3)\}$. Since the end point $p_3 = (3,3)$ of $P(t)$ and first point $q_0 = (3,3)$ of $Q(t)$ is same, therefore $P(t)$ and $Q(t)$ are joined. That is C^0-continuity exist at the joint.

By (1.13), we know that $P'(1) = n(p_n - p_{n-1})$ and $Q'(0) = n(q_1 - q_0)$, but $n = 3$, so

$$\begin{aligned} P'(1) &= 3(p_3 - p_2) = 3\{(3,3) - (2,1)\} = (3,6), \\ Q'(0) &= 3(q_1 - q_0) = 3\{(4,5) - (3,3)\} = (3,6). \end{aligned}$$

So $P'(1) = Q'(0) = (3,6)$. Hence $P(t)$ and $Q(t)$ are joined with C^1 continuity.

By (1.16),

$$\begin{aligned} P''(1) &= n(n-1)(p_n - 2p_{n-1} + p_{n-2}), \\ Q''(0) &= n(n-1)(q_2 - 2q_1 + q_0). \end{aligned}$$

For $n = 3$, we have

$$\begin{aligned} P''(1) &= 3(3-1)(p_3 - 2p_2 + p_1), \\ Q''(0) &= 3(3-1)(q_2 - 2q_1 + q_0). \end{aligned}$$

This implies

$$\begin{aligned} P''(1) &= 6\{(3,3) - 2(2,1) + (1,5)\} = (0,36), \\ Q''(0) &= 6\{(5,1) - 2(4,5) + (3,3)\} = (0,-36). \end{aligned}$$

Since $P''(1) \neq Q''(0)$, so $P(t)$ and $Q(t)$ are not joined with C^2 continuity. But

$$Q''(0) = (0,-36) = -1(0,36) = -1P''(1),$$

so $P(t)$ and $Q(t)$ are joined with G^2-continuity. □

Example 1.6.4. If following is the composite Be'zier curve

$$\begin{aligned} P(t) &= \{(-9,0),(-8,1),(-8,2.5),(-4,2.5)\}, \\ Q(t) &= \{(-4,2.5),(-3,3.5),(-4,4),(0,4)\}, \\ R(t) &= \{(0,4),(2,4),(3,4),(5,2)\}, \\ S(t) &= \{(5,2),(6,2),(20,3),(18,0)\}, \end{aligned}$$

then discuss the parametric and geometric continuities.

Solution. **C^0-, G^0-continuity:** Since

$$p_3 = q_0 = (-4,2.5), \quad q_3 = r_0 = (0,4), \quad r_3 = s_0 = (5,2),$$

therefore all cubic Be'zier curves are joined so composite Be'zier curve is C^0 as well as G^0-continuous.

C^1-, G^1-continuity:

$$\begin{aligned} P'(1) &= 3(p_3 - p_2) = 3[(-4,2.5) - (-8,2.5)] = (12,0), \\ Q'(0) &= 3(q_1 - q_0) = 3[(-3,3.5) - (-4,2.5)] = (3,4.5). \end{aligned}$$

As $P'(1) \neq Q'(0)$, so $P(t)$ and $Q(t)$ curves are not smoothly joint that is not joined with C^1 continuity. Here G^1-continuity also does not exist. Since

$$\begin{aligned} Q'(1) &= 3(q_3 - q_2) = 3[(0,4) - (-4,4)] = (12,0), \\ R'(0) &= 3(r_1 - r_0) = 3[(2,4) - (0,4)] = (6,0). \end{aligned}$$

As $Q'(1) \neq R'(0)$, so $Q(t)$ and $R(t)$ are not joined with C^1 continuity. But $Q'(1) = (12,0) = 2(6,0) = 2R'(0)$, so $Q(t)$ and $R(t)$ are joined with G^1 continuity. Since

$$\begin{aligned} R'(1) &= 3(r_3 - r_2) = 3[(5,2) - (3,4)] = (6,-6), \\ S'(0) &= 3(s_1 - s_0) = 3[(6,2) - (5,2)] = (3,0). \end{aligned}$$

As $R'(1) \neq S'(0)$, so $R(t)$ and $S(t)$ are not joined with C^1 continuity. Here G^1-continuity also does not exist. Here we conclude that the composite Be'zier curve is not C^1 as well not G^1-continuous on the entire domain. □

Definition 1.6.1. Affine and convex combinations of points: If $p_1, p_2, p_3, \cdots, p_n$ are the points and $\alpha_1, \alpha_2, \alpha_3, \cdots, \alpha_n$ are scalars such that $\alpha_1 + \alpha_2 + \alpha_3 + \cdots + \alpha_n = 1$, then

$$\alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3 + \cdots\cdots + \alpha_n p_n \tag{1.17}$$

is called affine combination of $p_1, p_2, p_3, \cdots, p_n$. Since $\alpha_1 = 1 - \alpha_2 - \alpha_3 - \cdots - \alpha_n$, therefore affine combination (1.17) can be written as

$$p_1 + \alpha_2(p_2 - p_1) + \alpha_3(p_3 - p_1) + \cdots + \alpha_n(p_n - p_1). \tag{1.18}$$

The combinations (1.17) and (1.18) are called convex combinations if

$$\alpha_1 + \alpha_2 + \alpha_3 + \cdots + \alpha_n = 1, \quad \text{and} \quad 0 \leq \alpha_i \leq 1, \quad \forall \quad i = 1, 2, 3, \cdots, n.$$

Geometrical interpretations:

Let p_1 and p_2 be any points then $p = \alpha_1 p_1 + \alpha_2 p_2$, where $\alpha_1 + \alpha_2 = 1$, and $0 \leq \alpha_1, \alpha_1, \leq 1$, is convex combination of p_1, p_2. Since $\alpha_1 = 1 - \alpha_2$, therefore $p = p_1 + \alpha_2(p_2 - p_1)$. The point p will always lie on the line segment joining the points p_1 and p_2. When

$\alpha_2 = 0$ then $p = p_1$ and when $\alpha_2 = 1$ then $p = p_2$. When $0 < \alpha_1, \alpha_1, < 1$, then point p will lie on the line segment in between the points p_1 and p_2.
If $p = p_1 + \alpha_2(p_2 - p_1) + \alpha_3(p_3 - p_1)$, and $\alpha_1 + \alpha_2 + \alpha_3 = 1$, with $0 \leq \alpha_1,\ \alpha_2,\ \alpha_3 \leq 1$, then the point p is called convex combination of the points p_1, p_2, and p_3.

- If $\alpha_i = 0$ for any $i = 1, 2, 3$ then p will be on the boundary of triangle made by the points p_1, p_2, and p_3,
- If $0 < \alpha_i < 1$ then p will be inside of the triangle,
- If any $\alpha_i < 0$ or $\alpha_i > 1$ then point p will be outside of the triangle.

Definition 1.6.2. Convex set: A subset C of plane is said to be a convex set provided that all the points on the line segment joining any two points in C are also elements of the set C (see Figure 1.1).

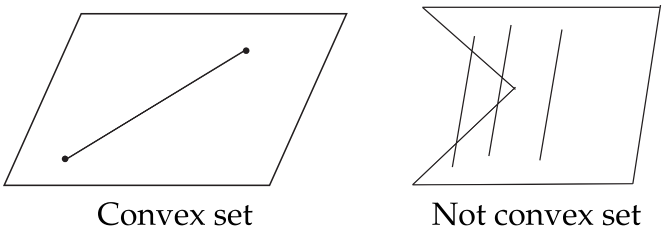


Figure 1.1: Convex and non-convex set

Definition 1.6.3. Convex Hull: The convex hull of the set C is the intersection of all convex sets contained in C. The convex hull is the smallest convex set containing a set of points. Geometrically, imagine the control points as being pegs, the convex hull of control points is a shape of a rubber-band stretched around the pegs (see Figure 1.2).
Mathematically, If $p_1, p_2, p_3, \cdots, p_n$ are the points and $\alpha_1, \alpha_2, \alpha_3, \cdots, \alpha_n$ are scalars and $\sum_{i=1}^{n} \alpha_i p_i$ is the linear combinations of points then the convex Hull property is satisfied if and only if

- All α_i are non negative, i.e. $\alpha_i \geqslant 0$, and $\sum_{i=1}^{n} \alpha_i = 1$.

Property 1.6.5. ***Be'zeir curve satisfies convex hull property***

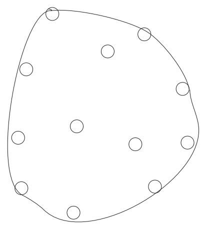


Figure 1.2: Convex Hull

We know that $P(t) = \sum_{i=0}^{n} B_{i,n}(t)p_i$ represents Be'zeir curve. Since $B_{i,n}(t)$ are non-negative, and $\sum_{i=0}^{n} B_{i,n}(t) = 1$, therefore Be'zeir curve satisfies the convex Hull property. It means, Be'zeir curve will lie inside the convex hull of the points $p_0, p_1, \cdots, p_n$.

Property 1.6.6. ***Be'zeir curve satisfies affine invariance property***

It is often necessary to subject to Be'zeir curve to an affine transformation in order to scale it, orient it, or position it for subsequent use. Suppose we wish to transform point $P(t)$ on the Be'zeir curve to the new point $Q(t)$, using the affine transform $T : A^m \to A^n$, i.e.

$$Q(t) = T(P(t)) = T\left(\sum_{i=0}^{n} B_{i,n}(t)p_i\right).$$

It means that, to find $Q(t)$ at any value of t, we must first evaluate $P(t)$ and then transform it, effectively starting over fresh for each new t. But this, in fact, is not so: we need only transform the control points (once) and then use the new control points in the same Bernstein form to re-create Be'zeir curve at any t that is

$$Q(t) = T(P(t)) = \sum_{i=0}^{n} T(p_i)B_{i,n}(t).$$

Affine invariance means that the transformed curve is identical to the curve that is based on the transformed control points.

Example 1.6.5. Figure 1.3 shows a Be'zeir curve based on the control points p_0, p_1, p_2, p_3. These points are rotated, scaled, and translated to the new control points q_0, q_1, q_2, q_3, and the Be'zeir curve determined by them is drawn. The curve is identical, point by point, to the result of transforming the original Be'zeir curve.

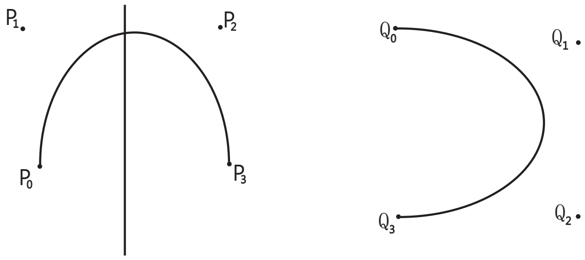


Figure 1.3: Affine invariance property

Property 1.6.7. ***Invariance under affine transformation of the parameter***

One may think that the Be'zeir curve is defined for the interval $[0, 1]$ i.e. $0 \leq t \leq 1$. This is done because it is convenient not because it is necessary. One may think of a curve as being defined on any arbitrary interval $[a, b]$ i.e. $a \leq u \leq b$. To do this, we simply put

$$t = \frac{u - a}{b - a}$$

and the transition from the interval $[0, 1]$ to $[a, b]$ is an affine map
when $u = a$, then $t = 0$,
when $u = b$, then $t = 1$.
It means Be'zeir curve defined over the interval $t \in [0, 1]$

$$P(t) = \sum_{i=0}^{n} B_{i,n}(t) p_i,$$

can be parameterized over the interval $u \in [a, b]$ as

$$P(u) = \sum_{i=0}^{n} B_{i,n}\left(\frac{u - a}{b - a}\right) p_i.$$

For example, if $u \in [1, 2]$ then

$$P(u) = \sum_{i=0}^{n} B_{i,n}\left(\frac{u - 1}{2 - 1}\right) p_i = \sum_{i=0}^{n} B_{i,n}(u - 1) p_i.$$

Property 1.6.8. ***Symmetry of Be'zeir curve***

It does not matter, if the control points of Be'zeir curve are labeled $p_0, p_1, p_2, p_3, \cdots, p_n$ or $p_n, p_{n-1}, \cdots, p_1, p_0$ in different ordering then Be'zeir curves are the same.

They differ only in the direction in which they were drawn i.e.

$$P(t) = \sum_{i=0}^{n} B_{i,n}(t)p_i = \sum_{i=0}^{n} B_{n-i,n}(1-t)p_i.$$

This follows from the identity

$$B_{n-i,n}(1-t) = \binom{n}{n-i}(1-t)^{n-i}(1-(1-t))^{n-(n-i)}.$$

This implies

$$B_{n-i,n}(1-t) = \binom{n}{n-i}(1-t)^{n-i}(t)^i = \binom{n}{i}t^i(1-t)^{n-i} = B_{i,n}(t).$$

Property 1.6.9. ***Be'zeir curve satisfies variation diminishing property***

Be'zeir curve cannot fluctuate more than their control polygon does, more precisely no straight line can have more intersection with Be'zier then it has with its control polygon. See Figure 1.4, where the straight line intersect the control polygon at 5-points but the Be'zier curve at 3-points.

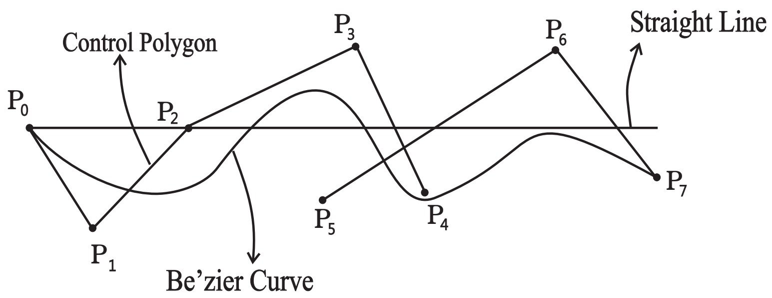


Figure 1.4: Variation diminishing property

Property 1.6.10. ***Linear precision***

The property of archiving a straight line / portion of a curve by properly positioning the control points is called linear precision. Can a Be'zier curve a straight line? Yes convex hull property shows that it can, if all the control points lie on the same line, the Be'zier curve be a straight line.

Property 1.6.11. ***Pseudo local control***

The Bernstein polynomial has only one maximum attain at $t = \frac{i}{n}$, this has a design application if we move only one of the central polygon vertex say p_i, then the curve is mostly effected by the change in the region of the curve around the perimetric value $\frac{i}{n}$, this make the effect of the change reasonably predictable. Although the change effects the whole curve but note that the curve always change globally (see Figure 1.5).

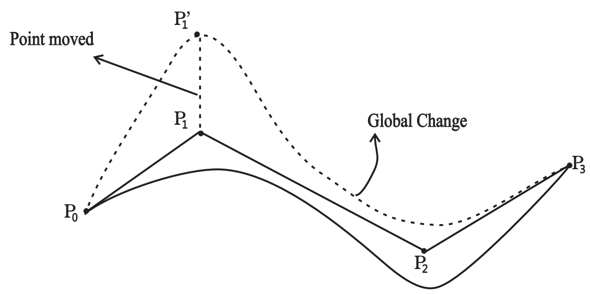


Figure 1.5: Pseudo local control

Definition 1.6.4. Support of the function The set of parameter t in which the basis function (i.e. Bernstein polynomial) is active (non-zero) is called support of the function.

1.7 The De-Casteljau algorithm

The De-Casteljau algorithm uses a sequence of points $p_0, p_1, p_2, \cdots, p_n$, construct a well defined value of $P(t)$ at each value of t from 0 to 1. Thus it provides a way to generate a curve from set of control points changing the point change the curve. For the given points we can construct a curve $P(t)$ by the following rule.

$$p_i^{(j)} = \begin{cases} (1-t)p_i^{(j-1)}(t) + tp_{i+1}^{j-1} & \text{if} \quad j > 0 \\ p_i & \text{if} \quad j = 0. \end{cases}$$

Example 1.7.1. Construct the curve from three points by using De-Casteljau algorithm.

Solution. Let us consider the control points p_0, p_1, p_2. Get a control polygon by joining these points with straight lines (see Figure 1.6a).

Step-I: First round (see Figure 1.6b)

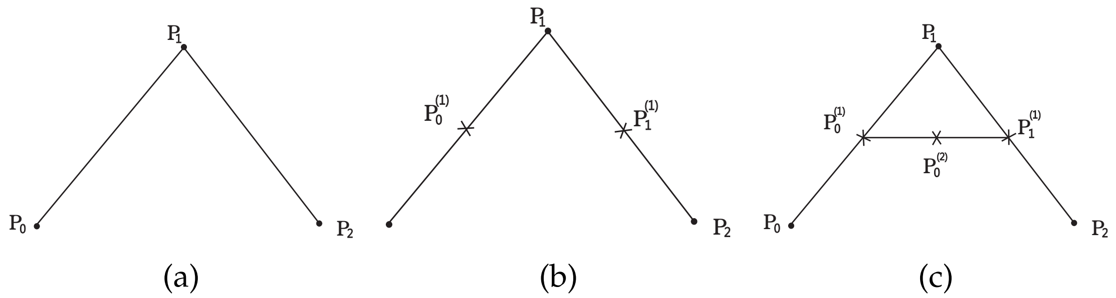


Figure 1.6: Step-I: Initial, 1st and 2nd rounds.

Compute the point $p_0^{(1)}$ on the segment p_0p_1

$$p_0^{(1)} = (1-t)p_0 + tp_1.$$

Compute the point $p_1^{(1)}$ on the segment p_1p_2

$$p_1^{(1)} = (1-t)p_1 + tp_2.$$

Second round: (see Figure 1.6c)

By joining $p_0^{(1)}$ and $p_1^{(1)}$, we get a new segment $p_0^{(1)}p_1^{(1)}$. Now find the point $p_0^{(2)}$ on the new segment by

$$p_0^{(2)} = (1-t)p_0^{(1)} + tp_1^{(1)}.$$

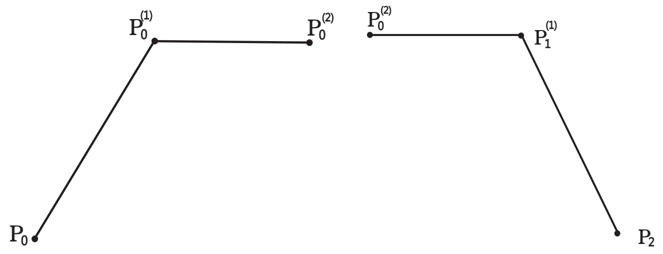
Step-II

Now divide the polygon into two sub-polygons i.e. the left and right polygons and apply the same procedure (see Figure 1.7).

Continue this procedure (see Figure 1.8), in the limit, we get smooth enough the left and right polygons. We get smooth Be'zier curve by joining these polygons.

□

Example 1.7.2. Develop quadratic Be'zier curve by using the reverse procedure of De-Casteljau algorithm.



(Left polygon) (Right polygon)

Figure 1.7: Step-II.

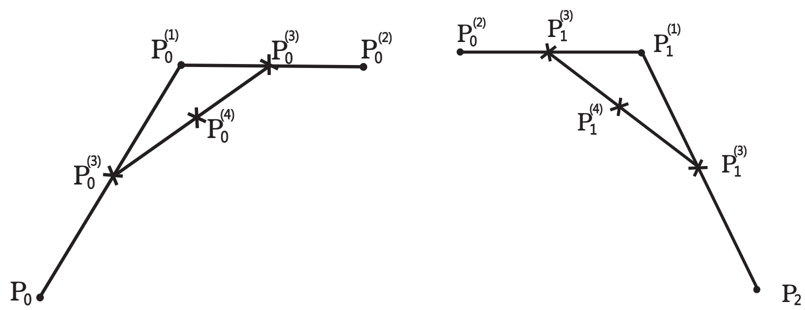


Figure 1.8: Step-II: 1st and 2nd rounds.

Solution. Start from the last point of De-Casteljau algorithm i.e.

$$\begin{aligned}
P(t) &= p_0^{(2)} \\
&= (1-t)p_0^{(1)} + tp_1^{(1)} \\
&= (1-t)[(1-t)p_0 + tp_1] + t[(1-t)p_1 + tp_2] \\
&= (1-t)^2 p_0 + 2t(1-t)p_1 + t^2 p_2 \\
&= p_0 B_{0,2}(t) + p_1 B_{1,2}(t) + p_2 B_{2,2}(t) \\
&= \sum_{i=0}^{2} p_i B_{i,2}(t),
\end{aligned}$$

which is a Be'zier curve of degree 2 i.e. quadratic Be'zier curve. □

Example 1.7.3. Construct the curve from four points by using De-Casteljau algorithm.

Solution. Let us consider the control points p_0, p_1, p_2, p_3. Get a control polygon by joining these points with straight lines (see Figure 1.9a).

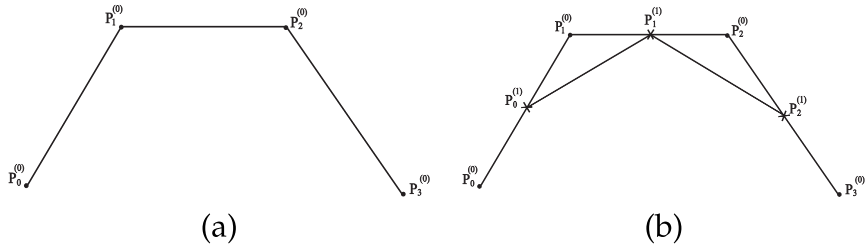


Figure 1.9: Step-I: Initial and first rounds.

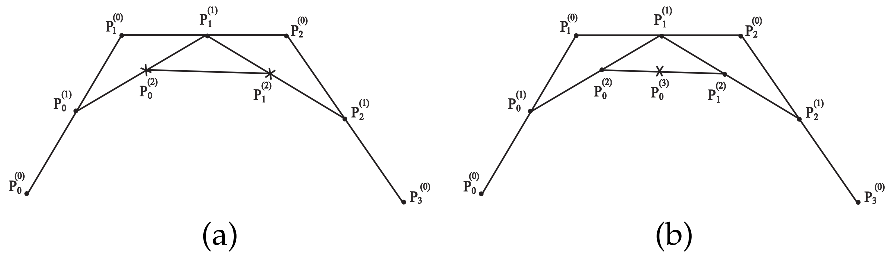


Figure 1.10: Step-I: 2nd and 3rd rounds.

Step-I: First round (see Figure 1.9b)

Compute the point $p_0^{(1)}$ on the segment p_0p_1

$$p_0^{(1)} = (1-t)p_0 + tp_1.$$

Compute the point $p_1^{(1)}$ on the segment p_1p_2

$$p_1^{(1)} = (1-t)p_1 + tp_2.$$

Compute the point $p_2^{(1)}$ on the segment p_2p_3

$$p_2^{(1)} = (1-t)p_2 + tp_3.$$

Second round: (see Figure 1.10a)

By joining $p_0^{(1)}$ and $p_1^{(1)}$, $p_1^{(1)}$ and $p_2^{(1)}$, we get a new segment $p_0^{(1)}p_1^{(1)}$, $p_1^{(1)}p_2^{(1)}$. Now find the point $p_0^{(2)}$ on the new segment by

$$p_0^{(2)} = (1-t)p_0^{(1)} + tp_1^{(1)}.$$

Now find the point $p_1^{(2)}$ on the new segment by

$$p_1^{(2)} = (1-t)p_1^{(1)} + tp_2^{(1)}.$$

Third round: (see Figure 1.10b)
By joining $p_0^{(2)}$ and $p_1^{(2)}$, we get a new segment $p_0^{(2)}p_1^{(2)}$. Now find the point $p_0^{(3)}$ on the new segment by

$$p_0^{(3)} = (1-t)p_0^{(2)} + tp_1^{(2)}.$$

Step-II
Now divide the polygon into two sub-polygons i.e. the left and right polygons and apply the same procedure.
Continue this procedure, in the limit, we get smooth enough the left and right polygons. We get smooth Be'zier curve by joining these polygons. □

Example 1.7.4. Develop cubic Be'zier curve by using the reverse procedure of De-Casteljau algorithm.

Solution. Start from the last point of De-Casteljau algorithm i.e.

$$\begin{aligned}
P(t) &= p_0^{(3)} \\
&= (1-t)p_0^{(2)} + tp_1^{(2)} \\
&= (1-t)[(1-t)p_0^{(1)} + tp_1^{(1)}] + t[(1-t)p_1^{(1)} + tp_2^{(1)}] \\
&= (1-t)^2 p_0^{(1)} + t(1-t)p_1^{(1)} + t(1-t)p_1^{(1)} + t^2 p_2^{(1)} \\
&= (1-t)^2 p_0^{(1)} + 2t(1-t)p_1^{(1)} + t^2 p_2^{(1)} \\
&= (1-t)^2[(1-t)p_0 + tp_1] + 2t(1-t)[(1-t)p_1 + tp_2] + t^2[(1-t)p_2 + tp_3] \\
&= (1-t)^3 p_0 + t(1-t)^2 p_1 + 2t(1-t)^2 p_1 + 2t^2(1-t)p_2 + t^2(1-t)p_2 + t^3 p_3 \\
&= (1-t)^3 p_0 + 3t(1-t)^2 p_1 + 3t^2(1-t)p_2 + t^3 p_3 \\
&= p_0 B_{0,3}(t) + p_1 B_{1,3}(t) + p_2 B_{2,3} + p_2 B_{3,3}(t) \\
&= \sum_{i=0}^{3} p_i B_{i,3}(t),
\end{aligned}$$

which is a cubic Be'zier curve. □

1.8 Matrix representation of Be′zier curve

Be′zier curves can easily be represented in matrix notations.

1.8.1 Matrix representation of linear Be′zier curve

The algebraic form of linear Be′zier curve (i.e. $n = 1$) is

$$P(t) \;=\; \sum_{i=0}^{1} p_i B_{i,1}(t) = p_0 B_{0,1}(t) + p_1 B_{1,1}(t) = (1-t)p_0 + tp_1.$$

This implies

$$P(t) = \begin{bmatrix} (1-t) & t \end{bmatrix} \begin{pmatrix} p_0 \\ p_1 \end{pmatrix}.$$

The matrix form of linear Bezier curve is

$$P(t) = \begin{pmatrix} 1 & t \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \end{pmatrix}.$$

1.8.2 Matrix representation of quadratic Be′zier curve

The algebraic form of quadratic Be′zier curve (i.e. $n = 2$) is

$$\begin{aligned} P(t) &= \sum_{i=0}^{2} p_i B_{i,2}(t) = p_0 B_{0,2}(t) + p_1 B_{1,2}(t) + p_2 B_{2,2}(t) \\ &= (1-t)^2 p_0 + 2t(1-t)p_1 + t^2 p_2. \end{aligned}$$

This implies

$$P(t) = \begin{bmatrix} (1-t)^2 & 2t(1-t) & t^2 \end{bmatrix} \begin{pmatrix} p_0 \\ p_1 \\ p_2 \end{pmatrix}.$$

This can be written as

$$P(t) = \begin{pmatrix} 1-2t+t^2 & 2t-2t^2 & t^2 \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ p_2 \end{pmatrix}.$$

The matrix form of quadratic Bezier curve is

$$P(t) = \begin{pmatrix} 1 & t & t^2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -2 & 2 & 0 \\ 1 & -2 & 1 \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ p_2 \end{pmatrix}.$$

1.8.3 Matrix representation of cubic Be′zier curve

The algebraic form of cubic Be′zier curve for $t \in [0, 1]$ is

$$\begin{aligned} P(t) &= \sum_{i=0}^{3} p_i B_{i,3}(t) = p_0 B_{0,3}(t) + p_1 B_{1,3}(t) + p_2 B_{2,3}(t) + p_3 B_{3,3}(t) \\ &= (1-t)^3 p_0 + 3t(1-t)^2 p_1 + 3t^2(1-t) p_2 + t^3 p_3. \end{aligned}$$

This implies

$$P(t) = \begin{pmatrix} (1-t)^3 & 3t(1-t)^2 & 3t^2(1-t) & t^3 \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix}.$$

Again implies

$$P(t) = \begin{pmatrix} 1+3t-3t^2-t^3 & 3t-6t^2+3t^3 & 3t^2-3t^3 & t^3 \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix}.$$

The matrix form of cubic Bezier curve is

$$P(t) = \begin{pmatrix} 1 & t & t^2 & t^3 \end{pmatrix} M \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix}, \tag{1.19}$$

where

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{pmatrix}. \tag{1.20}$$

Example 1.8.1. Find the control points or control polygon for the portion of the curve for $t \in [0, 1/2]$ also find the cubic Be'zier curve.

Solution. Since in general, we define Be'zier curve over the domain $t \in [0, 1]$, but the problem under discussion has domain $[0, 1/2] = [a, b]$, so use parametrization i.e. replace t by $(b-a)t + a = (1/2 - 0)t + 0 = t/2$. This means that the cubic Be'zier curve $Q(t)$ over the domain $[0, 1/2]$ can be obtained by replacing t by $t/2$ in (1.19).

$$Q(t) = P(t/2) = \begin{bmatrix} 1 & t/2 & t^2/4 & t^3/8 \end{bmatrix} M \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix}.$$

This implies

$$Q(t) = \begin{bmatrix} 1 & t & t^2 & t^3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/4 & 0 \\ 0 & 0 & 0 & 1/8 \end{bmatrix} M \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix}.$$

This again implies

$$Q(t) \;=\; \begin{bmatrix} 1 & t & t^2 & t^3 \end{bmatrix} M M^{-1} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/4 & 0 \\ 0 & 0 & 0 & 1/8 \end{bmatrix} M \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix}.$$

This implies

$$Q(t) \;=\; \begin{bmatrix} 1 & t & t^2 & t^3 \end{bmatrix} M \, S_{[0,\frac{1}{2}]} \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix}, \tag{1.21}$$

where

$$S_{[0,\frac{1}{2}]} = M^{-1} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/4 & 0 \\ 0 & 0 & 0 & 1/8 \end{bmatrix} M,$$

and

$$M \;=\; \begin{pmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{pmatrix}, \quad M^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1/3 & 0 & 0 \\ 1 & 2/3 & 1/3 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

This implies

$$S_{[0,\frac{1}{2}]} \;=\; \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 1/4 & 1/2 & 1/4 & 0 \\ \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8} \end{bmatrix}.$$

The control points for the portion of the curve for $t \in [0, 1/2]$ are given below

$$S_{[0,\frac{1}{2}]}\begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} p_0 \\ 1/2p_0 + 1/2p_1 \\ 1/4p_0 + 1/2p_1 + 1/4p_2 \\ 1/8p_0 + 3/8p_1 + 3/8p_2 + 1/8p_3 \end{bmatrix},$$

By substituting it in (1.21), we get the cubic Be'zier curve over the interval $[0, 1/2]$.

$$Q(t) = \begin{bmatrix} 1 & t & t^2 & t^3 \end{bmatrix} M \begin{bmatrix} p_0 \\ 1/2p_0 + 1/2p_1 \\ 1/4p_0 + 1/2p_1 + 1/4p_2 \\ 1/8p_0 + 3/8p_1 + 3/8p_2 + 1/8p_3 \end{bmatrix}.$$

□

Example 1.8.2. Find the control points or control polygon for the portion of the curve for $t \in [1/2, 1]$ also find the cubic Be'zier curve.

Solution. The problem under discussion has domain $[1/2, 1] = [a, b]$, so use parametrization i.e. replace t by $(b - a)t + a = (1 - 1/2)t + 1/2 = (1/2)t + 1/2$. This means that the cubic Be'zier curve $Q(t)$ over the domain $[1/2, 1]$ can be obtained by replacing t by $(1/2)t + 1/2$ in (1.19).

$$\begin{aligned} Q(t) &= P[(1/2)t + 1/2] \\ &= \left[1, \quad 1/2 + t/2, \quad 1/4 + t/2 + t^2/4, \quad 1/8 + 3t/8 + 3t^2/8 + t^3/8\right] M \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix}. \end{aligned}$$

This implies

$$Q(t) = \begin{bmatrix} 1 & t & t^2 & t^3 \end{bmatrix} M M^{-1} \begin{bmatrix} 1 & 1/2 & 1/4 & 1/8 \\ 0 & 1/2 & 1/2 & 3/8 \\ 0 & 0 & 1/4 & 3/8 \\ o & 0 & 0 & 1/8 \end{bmatrix} M \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix}.$$

This implies

$$Q(t) = \begin{bmatrix} 1 & t & t^2 & t^3 \end{bmatrix} M S_{[1/2,1]} \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix}, \tag{1.22}$$

where

$$S_{[\frac{1}{2},1]} = M^{-1} \begin{bmatrix} 1 & 1/2 & 1/4 & 1/8 \\ 0 & 1/2 & 1/2 & 3/8 \\ 0 & 0 & 1/4 & 3/8 \\ 0 & 0 & 0 & 1/8 \end{bmatrix} M,$$

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix}, \quad M^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1/3 & 0 & 0 \\ 1 & 2/3 & 1/3 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

This further implies

$$\begin{aligned} S_{[\frac{1}{2},1]} &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1/3 & 0 & 0 \\ 1 & 2/3 & 1/3 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1/2 & 1/4 & 1/8 \\ 0 & 1/2 & 1/2 & 3/8 \\ 0 & 0 & 1/4 & 3/8 \\ 0 & 0 & 0 & 1/8 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1/8 & 3/8 & 3/8 & 1/8 \\ 0 & 1/4 & 1/2 & 1/4 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

The set of control points for the portion of the curve for $t \in [1/2, 1]$ is defined below

$$S_{[\frac{1}{2},1]}\begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} p_0/8 + 3p_1/8 + 3p_2/8 + p_3/8 \\ p_1/4 + p_2/2 + p_3/4 \\ p_2/2 + p_3/2 \\ p_3 \end{bmatrix},$$

while by substituting these points in (1.22), we get the cubic Be′zier curve

$$Q(t) = \begin{bmatrix} 1 & t & t^2 & t^3 \end{bmatrix} M \begin{bmatrix} p_0/8 + 3p_1/8 + 3p_2/8 + p_3/8 \\ p_1/4 + p_2/2 + p_3/4 \\ p_2/2 + p_3/2 \\ p_3 \end{bmatrix}.$$

□

Example 1.8.3. Find the control points or control polygon for the portion of the curve for $t \in [1, 2]$ also find the cubic Be′zier curve.

Solution. The problem under discussion has domain $[1, 2] = [a, b]$, so use parametrization i.e. replace t by $(b - a)t + a = (2 - 1)t + 1 = t + 1$. This means that the cubic Be′zier curve $Q(t)$ over the domain $[1, 2]$ can be obtained by replacing t by $t + 1$ in (1.19).

$$Q(t) = \begin{bmatrix} 1 & (1+t) & (1+t)^2 & (1+t)^3 \end{bmatrix} M \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix}.$$

This implies

$$Q(t) = \begin{bmatrix} 1 & 1+t & 1+2t+t^2 & 1+3t+3t^2+t^3 \end{bmatrix} M \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix}.$$

This further implies

$$Q(t) \;=\; \begin{bmatrix} 1 & t & t^2 & t^3 \end{bmatrix} M M^{-1} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ o & 0 & 0 & 1 \end{bmatrix} M \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix}.$$

This again implies

$$Q(t) \;=\; \begin{bmatrix} 1 & t & t^2 & t^3 \end{bmatrix} M S_{[1,2]} \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix}, \tag{1.23}$$

where

$$S_{[1,2]} = M^{-1} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} M,$$

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix}, \quad M^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1/3 & 0 & 0 \\ 1 & 2/3 & 1/3 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

$$S_{[1,2]} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1/3 & 0 & 0 \\ 1 & 2/3 & 1/3 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ o & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & -4 & 4 \\ -1 & 6 & -12 & 8 \end{bmatrix}.$$

The set of control points for the portion of the curve for $t \in [,2]$ is defined below

$$S_{[1,2]} \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} p_3 \\ -p_2 + 2p_3 \\ p_1 - 4p_2 + 4p_3 \\ -p_0 + 6p_1 - 12p_2 + 8p_3 \end{bmatrix},$$

while by substituting these points in (1.23), we get the cubic Be'zier curve

$$Q(t) = \begin{bmatrix} 1 & t & t^2 & t^3 \end{bmatrix} M \begin{bmatrix} p_3 \\ -p_2 + 2p_3 \\ p_1 - 4p_2 + 4p_3 \\ -p_0 + 6p_1 - 12p_2 + 8p_3 \end{bmatrix}.$$

□

Note: From the above discussion, we conclude that, if we apply $S^i_{[1,2]} S^k_{[0,1/2]}$, we obtain the Bezier control polygon for the portion of the curve where t ranges between $i/2^k$ and $(i+1)/2^k$.

1.9 Degree elevation

Suppose we were designing with Be'zier curve of degree n after an attempt, it may turned out that a degree n curve does not passes sufficient flexibility to model/describe shape. One way to proceed in such a situation is to increase the flexibility of polygon by adding another vertex to it. This corresponds to raise the degree of the polynomial by 1.

We are thus looking for a curve with vertices. $p_0^{(1)}, p_1^{(1)}, p_2^{(1)}, \cdots\cdots, p_{n+1}^{(1)}$, that describes the same curve as the original polygon $p_0, p_1, p_2, \cdots\cdots, p_n$. Let us describe our curve as

$$P(t) = (1-t)P(t) + tP(t). \tag{1.24}$$

Consider

$$\begin{aligned}
(1-t)P(t) &= (1-t)\sum_{i=0}^{n} p_i B_{i,n}(t) = \sum_{i=0}^{n}(1-t)\binom{n}{i}t^i(1-t)^{n-i}p_i \\
&= \sum_{i=0}^{n} \frac{n!}{i!(n-i)!}t^i(1-t)^{n-i+1}p_i \\
&= \sum_{i=0}^{n} \frac{(n+1)n!}{i!(n-i)!(n+1-i)}t^i(1-t)^{n+1-i}\left(\frac{n+1-i}{n+1}\right)p_i \\
&= \sum_{i=0}^{n} \frac{(n+1)!}{i!(n+1-i)!}t^i(1-t)^{n+1-i}\left(\frac{n+1-i}{n+1}\right)p_i \\
&= \sum_{i=0}^{n} \binom{n+1}{i}t^i(1-t)^{n+1-i}\left(\frac{n+1-i}{n+1}\right)p_i \\
&= \sum_{i=0}^{n} B_{i,n+1}(t)\left(\frac{n+1-i}{n+1}\right)p_i.
\end{aligned}$$

This implies

$$(1-t)P(t) = \sum_{i=0}^{n+1} B_{i,n+1}(t)p_i\left(\frac{n+1-i}{n+1}\right),$$

because when $i = n+1$ then $\frac{n+1-i}{n+1} = 0$, so adding $(n+1)$th term which is zero, does not make any change in the summation.

Now consider

$$\begin{aligned}
tP(t) &= t\sum_{i=0}^{n} p_i B_{i,n}(t) = t\sum_{i=0}^{n}\binom{n}{i} t^i(1-t)^{n-i}p_i \\
&= \sum_{i=0}^{n}\binom{n}{i} t^{i+1}(1-t)^{n-i}p_i = \sum_{i=0}^{n}\frac{n!}{i!(n-i)!}t^{i+1}(1-t)^{n-i}p_i \\
&= \sum_{i=0}^{n}\frac{n!(n+1)}{i!(n-i)!(i+1)}t^{i+1}(1-t)^{n-i}\left(\frac{i+1}{n+1}\right)p_i \\
&= \sum_{i=0}^{n}\frac{(n+1)!}{(i+1)!(n-i)!}t^{i+1}(1-t)^{n-i}\left(\frac{i+1}{n+1}\right)p_i \\
&= \sum_{i=0}^{n}\binom{n+1}{i+1} t^{i+1}(1-t)^{n-i}\left(\frac{i+1}{n+1}\right)p_i \\
&= \sum_{i=0}^{n} B_{i+1,n+1}(t)\left(\frac{i+1}{n+1}\right)p_i.
\end{aligned}$$

Replace i by $i-1$,

$$tP(t) = \sum_{i-1=0}^{i-1=n} B_{(i-1)+1,n+1}(t)p_{i-1}\left(\frac{(i-1)+1}{n+1}\right).$$

This implies

$$tP(t) = \sum_{i=1}^{i=n+1} B_{i,n+1}(t)p_{i-1}\left(\frac{i}{n+1}\right).$$

This again implies

$$tP(t) = \sum_{i=0}^{n+1} B_{i,n+1}(t)p_{i-1}\left(\frac{i}{n+1}\right),$$

because when $i=0$ then $\frac{i}{n+1}=0$, so adding 0th term which is zero, does not make any change in the summation. Now by substituting the values of terms $(1-t)P(t)$ and $tP(t)$ in (1.24), we get

$$\begin{aligned}
P(t) &= \sum_{i=0}^{n+1} B_{i,n+1}(t)p_i\left(\frac{n+1-i}{n+1}\right) + \sum_{i=0}^{n+1} B_{i,n+1}(t)p_{i-1}\left(\frac{i}{n+1}\right), \\
&= \sum_{i=0}^{n+1}\left[\left(\frac{n+1-i}{n+1}\right)p_i + \left(\frac{i}{n+1}\right)p_{i-1}\right]B_{i,n+1}(t).
\end{aligned}$$

So we have raised the degree by 1, that is degree of Be'zier curve is $n+1$ because there were $n+1$ points in the beginning, now there are $n+2$ points, that is for $i = 0, 1, 2, \cdots, n+1$

$$p_i^{(1)} = \left(1 - \frac{i}{n+1}\right) p_i + \left(\frac{i}{n+1}\right) p_{i-1}. \tag{1.25}$$

1.9.1 Repeated degree elevation

This process of degree elevation assigns a polygon $p^{(i)}$ to original polygon p. We may repeat this process and obtained sequence of polygons.

$$p = p^0, p^{(1)}, p^{(2)}, p^{(3)}, \cdots .$$

After r times degree elevation the polygon $p^{(r)}$ has the vertices

$$p_0^r, p_1^r, p_2^r, \cdots p_{n+r}^r$$

and each $p_i^{(r)}$ is explicitly given by for $i = 0, 1, 2, 3, \cdots, n+r$

$$p_i^{(r)} = \sum_{j=0}^{n} p_j \binom{n}{j} \binom{r}{i-j} \Big/ \binom{n+r}{i} \tag{1.26}$$

where

$$\begin{aligned} n &= \text{degree of curve (before the degree elevation)}, \\ r &= \text{how much degree is raised or number of steps.} \end{aligned}$$

Example 1.9.1. If we have Be'zier curve of degree 3 then raise its degree 2 times.

Solution. Consider the Be'zier curve of degree $n = 3$

$$P(t) = \sum_{i=0}^{3} p_i B_{i,3}(t).$$

After one time degree raising, we will get

$$P(t) = \sum_{i=0}^{4} p_i^{(1)} B_{i,4}(t),$$

where for $n = 3$ and by (1.26), we have the following control points of Be'zier curve of degree 4

$$p_i^{(1)} = \left(1 - \frac{i}{3+1}\right) p_i + \left(\frac{i}{3+1}\right) p_{i-1}, \quad i = 0, 1, 2, 3, 4.$$

For $i = 0, 1, 2, 3, 4$

$$\begin{aligned} p_0^{(1)} &= p_0, \\ p_1^{(1)} &= \frac{3}{4}p_1 + \frac{1}{4}p_0, \\ p_2^{(1)} &= \frac{1}{2}p_2 + \frac{1}{2}p_1, \\ p_3^{(1)} &= \frac{1}{4}p_3 + \frac{3}{4}p_2, \\ p_4^{(1)} &= p_3. \end{aligned}$$

After 2nd time degree raising, we will get Be'zier curve of degree 5

$$P(t) = \sum_{i=0}^{5} p_i^{(2)} B_{i,5}(t),$$

where for $n = 4$ and by (1.26), following are the control points of Be'zier curve of degree 5

$$p_i^{(2)} = \left(1 - \frac{i}{4+1}\right) p_i^{(1)} + \left(\frac{i}{4+1}\right) p_{i-1}^{(1)}, \quad i = 0, 1, 2, 3, 4, 5.$$

For $i = 0, 1, 2, 3, 4, 5$

$$
\begin{aligned}
p_0^{(2)} &= p_0^{(1)}, \\
p_1^{(2)} &= \frac{4}{5}p_1^{(1)} + \frac{1}{5}p_0^{(1)}, \\
p_2^{(2)} &= \frac{3}{5}p_2^{(1)} + \frac{2}{5}p_1^{(1)}, \\
p_3^{(2)} &= \frac{2}{5}p_3^{(1)} + \frac{3}{5}p_2^{(1)}, \\
p_4^{(2)} &= \frac{1}{5}p_4^{(1)} + \frac{4}{5}p_3^{(1)}, \\
p_5^{(2)} &= p_4^{(1)}.
\end{aligned}
$$

This implies

$$
\begin{aligned}
p_0^{(2)} &= p_0, \\
p_1^{(2)} &= \frac{4}{5}\left(\frac{3}{4}p_1 + \frac{1}{4}p_0\right) + \frac{1}{5}p_0 = \frac{3}{5}p_1 + \frac{2}{5}p_0, \\
p_2^{(2)} &= \frac{3}{5}\left(\frac{1}{2}p_2 + \frac{1}{2}p_1\right) + \frac{2}{5}\left(\frac{3}{4}p_1 + \frac{1}{4}p_0\right) = \frac{3}{10}p_2 + \frac{6}{10}p_1 + \frac{1}{10}p_0, \\
p_3^{(2)} &= \frac{2}{5}\left(\frac{1}{4}p_3 + \frac{3}{4}p_2\right) + \frac{3}{5}\left(\frac{1}{2}p_2 + \frac{1}{2}p_1\right) = \frac{1}{10}p_3 + \frac{6}{10}p_2 + \frac{3}{10}p_1, \\
p_4^{(2)} &= \frac{1}{5}p_3 + \frac{4}{5}\left(\frac{1}{4}p_3 + \frac{3}{4}p_2\right) = \frac{2}{5}p_3 + \frac{3}{5}p_3, \\
p_5^{(2)} &= p_3.
\end{aligned}
$$

By explicit method:

Now we raise the degree by two by using explicit method. That is put $n = 3$ and $r = 2$ in (1.26), we directly get the following control points of Be'zier curve of degree 5.

$$
p_i^{(2)} = \sum_{j=0}^{3} p_j \binom{3}{j}\binom{2}{i-j} \Big/ \binom{3+2}{i}, \quad i = 0, 1, 2, 3, \cdots, 3+2 = 5,
$$

By substituting $i = 0, 1, 2, 3, 4, 5$, we get

$$
\begin{aligned}
p_0^{(2)} &= \sum_{j=0}^{3} p_j \binom{3}{j}\binom{2}{0-j} \Big/ \binom{5}{0} \\
&= p_0\binom{3}{0}\binom{2}{0} \Big/ \binom{5}{0} + p_1\binom{3}{1}\binom{2}{-1} \Big/ \binom{5}{0} \\
&\quad + p_2\binom{3}{2}\binom{2}{-2} \Big/ \binom{5}{0} + p_3\binom{3}{3}\binom{2}{-3} \Big/ \binom{5}{0} \\
&= p_0 + 0 + 0 + 0 = p_0,
\end{aligned}
$$

$$
\begin{aligned}
p_1^{(2)} &= \sum_{j=0}^{3} p_j \binom{3}{j}\binom{2}{1-j} \Big/ \binom{5}{1} \\
&= p_0\binom{3}{0}\binom{2}{1} \Big/ \binom{5}{1} + p_1\binom{3}{1}\binom{2}{0} \Big/ \binom{5}{1} \\
&\quad + p_2\binom{3}{2}\binom{2}{-1} \Big/ \binom{5}{1} + p_3\binom{3}{3}\binom{2}{-2} \Big/ \binom{5}{1} \\
&= p_0(1)(2)/5 + p_1(3)(1)/5 + 0 + 0 + 0 \\
&= \frac{2}{5}p_0 + \frac{3}{5}p_1,
\end{aligned}
$$

$$
\begin{aligned}
p_2^{(2)} &= \sum_{j=0}^{3} p_j \binom{3}{j}\binom{2}{2-j} \Big/ \binom{5}{2} \\
&= p_0\binom{3}{0}\binom{2}{2} \Big/ \binom{5}{2} + p_1\binom{3}{1}\binom{2}{1} \Big/ \binom{5}{2} \\
&\quad + p_2\binom{3}{2}\binom{2}{0} \Big/ \binom{5}{2} + p_3\binom{3}{3}\binom{2}{-1} \Big/ \binom{5}{2} \\
&= \frac{1}{10}p_0 + p_1(3)(2)/10 + p_2(3)(1)/10 \\
&= \frac{1}{10}p_0 + \frac{6}{10}p_1 + \frac{3}{10}p_2,
\end{aligned}
$$

$$\begin{aligned}
p_3^{(2)} &= \sum_{j=0}^{3} p_j \binom{3}{j}\binom{2}{3-j} \Big/ \binom{5}{3} \\
&= p_0\binom{3}{0}\binom{2}{3} \Big/ \binom{5}{3} + p_1\binom{3}{1}\binom{2}{2} \Big/ \binom{5}{3} \\
&\quad + p_2\binom{3}{2}\binom{2}{1} \Big/ \binom{5}{3} + p_3\binom{3}{3}\binom{2}{0} \Big/ \binom{5}{3} \\
&= p_0(0) + p_1(3)/10 + p_2(3)(2)/10 + p_3(1)(1)/10 \\
&= \frac{3}{10}p_1 + \frac{6}{10}p_2 + \frac{1}{10}p_3,
\end{aligned}$$

$$\begin{aligned}
p_4^{(2)} &= \sum_{j=0}^{3} p_j \binom{3}{j}\binom{2}{4-j} \Big/ \binom{5}{4} \\
&= p_0\binom{3}{0}\binom{2}{4} \Big/ \binom{5}{4} + p_1\binom{3}{1}\binom{2}{3} \Big/ \binom{5}{4} \\
&\quad + p_2\binom{3}{2}\binom{2}{2} \Big/ \binom{5}{4} + p_3\binom{3}{3}\binom{2}{1} \Big/ \binom{5}{4} \\
&= p_0(0) + p_1(0) + p_2(3)(1)/5 + p_3(1)(2)/5 \\
&= \frac{3}{5}p_2 + \frac{2}{5}p_3,
\end{aligned}$$

$$\begin{aligned}
p_5^{(2)} &= \sum_{j=0}^{3} p_j \binom{3}{j}\binom{2}{5-j} \Big/ \binom{5}{5} \\
&= p_0\binom{3}{0}\binom{2}{5} \Big/ \binom{5}{5} + p_1\binom{3}{1}\binom{2}{4} \Big/ \binom{5}{5} \\
&\quad + p_2\binom{3}{2}\binom{2}{3} \Big/ \binom{5}{5} + p_3\binom{3}{3}\binom{2}{2} \Big/ \binom{5}{5} \\
&= p_0(0) + p_1(0) + p_2(0) + p_3(1)(1)/1 \\
&= p_3.
\end{aligned}$$

□